A NEW FRACTIONAL DERIVATIVE MODEL FOR THE ANOMALOUS DIFFUSION PROBLEM

by

Zhanqing CHEN [a,b], ***Peitao QIU*** [a,c*], ***Xiao-Jun YANG*** [a*], ***Yiying FENG*** [b], **and** ***Jiangen LIU*** [d]

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, Jiangsu, China
[b] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou, Jiangsu, China
[c] School of Civil Engineering, Xuzhou University of Technology, Xuzhou, China
[c] School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, China

Original scientific paper
https://doi.org/10.2298/TSCI180912253C

In this paper, a new fractional derivative within the exponential decay kernel is addressed for the first time. A new anomalous diffusion model is proposed to describe the heat-conduction problem. With the use of the Laplace transform, the analytical solution is discussed in detail. The presented result is as an accurate and efficient approach proposed for the heat-conduction problem in the complex phenomena.

Key words: *fractional derivative, exponential decay kernel, anomalous diffusion, analytical solution, Laplace transform*

Introduction

The theory of the fractional calculus (FC) has successfully utilized to describe the mathematical problems in physics and applied science, [1-4]. The time-fractional anomalous diffusion model, as one of the important applications in FC, was investigated by many scientists. For example, the time-fractional diffusion model for the kinetic description of anomalous diffusion and relaxation phenomena was reported in [5]. The numerical solution of the time-fractional diffusion equation was presented [6]. The continuous time random walks for the time-fractional diffusion problem were proposed [7]. The Wright-type solution of the time-fractional diffusion equation was suggested [8]. The numerical solution of the time-space fractional diffusion model with the aid of the matrix transfer technique was represented [9].

The classical fractional derivatives involving the singular kernel in the bounded domain were proposed by Liouville [10], Riemann [11], Sonine [12], and Caputo [13]. The general derivatives within the exponential kernel were proposed [4, 14, 15]. By the motivation of these results, the main aims of the papers are to propose the new fractional derivative within the exponential decay kernel based on the Liouville-Sonine and Liouville-Sonine-Caputo fractional derivatives and the general derivatives and to present the new anomalous diffusion model.

[*] Corresponding author, e-mail: qpt198959@163.com; dyangxiaojun@163.com

A new fractional derivative within the exponential decay kernel

In this section, based on the classical fractional derivatives and the general derivatives, we propose the new fractional derivative within the exponential decay kernel.

Let $\mathbb{N}$ and $\mathbb{N}_0$ be the set of the natural numbers and $\mathbb{N}_0 = \mathbb{N}_0 \cup \{0\}$, respectively.

Definition 1. Let $0 \le \alpha < 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$. The left-sided Liouville-Sonine fractional derivative is defined [1, 4, 10, 12]:

$$ {}_{\mathrm{LS}}\mathrm{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{(1)}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \tag{1} $$

and the right-sided Liouville-Sonine fractional derivative [1, 4, 10, 12]:

$$ {}_{\mathrm{LS}}\mathrm{D}_{b-}^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b \frac{f^{(1)}(\tau)}{(\tau-t)^{\alpha}} \mathrm{d}\tau \tag{2} $$

Definition 2. Let $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$. The left-sided Liouville-Sonine-Caputo fractional derivative is defined [1, 4, 10, 12, 13]:

$$ {}_{\mathrm{LSC}}\mathrm{D}_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\kappa-\alpha)} \int_a^t \frac{f^{(\kappa)}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \tag{3} $$

and the right-sided Liouville-Sonine-Caputo fractional derivative [1, 4]:

$$ {}_{\mathrm{LSC}}\mathrm{D}_{b-}^{\alpha} f(x) = \frac{(-1)^{\kappa}}{\Gamma(\kappa-\alpha)} \int_t^b \frac{f^{(\kappa)}(\tau)}{(\tau-t)^{\alpha}} \mathrm{d}\tau \tag{4} $$

Definition 3. Let $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$. The left-sided Liouville fractional derivative is defined [1, 10]:

$$ {}_{\mathrm{LSC}}\mathrm{D}_{+}^{\alpha} f(x) = \frac{1}{\Gamma(\kappa-\alpha)} \int_{-\infty}^t \frac{f^{(\kappa)}(\tau)}{(t-\tau)^{\alpha-\kappa+1}} \mathrm{d}\tau \tag{5} $$

and the right-sided Liouville fractional derivative [4]:

$$ {}_{\mathrm{LSC}}\mathrm{D}_{-}^{\alpha} f(x) = \frac{(-1)^{\kappa}}{\Gamma(\kappa-\alpha)} \int_t^{\infty} \frac{f^{(\kappa)}(\tau)}{(\tau-t)^{\alpha-\kappa+1}} \mathrm{d}\tau \tag{6} $$

Definition 4. Let $0 \le \alpha < 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$ and $\lambda > 0$. The left-sided Caputo-Fabrizio general derivative within the exponential decay kernel is defined [14]. For more details see [4, 15]:

$$ {}_{\mathrm{R}}^{\mathrm{CF}}\mathrm{D}_{a+}^{\alpha,\lambda} f(t) = J(\alpha) \int_a^t \exp\left[-\lambda(t-\tau)\right] f^{(\kappa)}(\tau) \mathrm{d}\tau \tag{7} $$

and the right-sided general derivative within the exponential decay kernel:

$$ {}_{\mathrm{R}}^{\mathrm{CF}}\mathrm{D}_{b-}^{\alpha,\lambda} f(t) = -J(\alpha) \int_t^b \exp\left[-\lambda(\tau-t)\right] f^{(\kappa)}(\tau) \mathrm{d}\tau \tag{8} $$

where

$$J(\alpha)=\frac{(2-\alpha)\Im(\alpha)}{2(1-\alpha)} \tag{9}$$

and

$$\lambda=\frac{\alpha}{1-\alpha} \tag{10}$$

with the normalization constant $\Im(\alpha)$, where $\Im(1) = \Im(0) = 1$.

Definition 5. Let $\kappa \geq 0$, $\kappa \in \mathbb{N}_0$, $-\infty < a < b < \infty$ and $\lambda > 0$. The left-sided general derivative within the exponential decay kernel is defined [4]:

$$\mathrm{D}_{a+}^{\kappa,\lambda} f(t)=\int_a^t \exp\left[-\lambda(t-\tau)\right] f^{(\kappa)}(\tau)\mathrm{d}\tau \tag{13}$$

and the right-sided general derivative within the exponential decay kernel is defined [4]:

$$\mathrm{D}_{b-}^{\kappa,\lambda} f(t)=(-1)^{\kappa}\int_t^b \exp\left[-\lambda(\tau-t)\right] f^{(\kappa)}(\tau)\mathrm{d}\tau \tag{14}$$

Definition 6. Let $\kappa \leq \alpha \leq \kappa + 1$, $\kappa \geq 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$. and $\lambda > 0$. The left-sided fractional derivative within the exponential decay kernel is defined:

$$\begin{aligned}&\mathrm{D}_{a+}^{\alpha,\lambda} f(t)=\\&=\int_a^t \exp\left[-\lambda(t-\tau)\right]\left[{}_{\mathrm{LSC}}\mathrm{D}_{a+}^{\alpha} f(\tau)\right]\mathrm{d}\tau=\\&=\int_a^t \exp\left[-\lambda(\varsigma-\xi)\right]\left[\frac{1}{\Gamma(\kappa-\alpha)}\int_a^{\varsigma}\frac{f^{(\kappa)}(\tau)}{(\varsigma-\tau)^{\alpha-\kappa+1}}\mathrm{d}\tau\right]\mathrm{d}\xi\end{aligned} \tag{15}$$

and the right-sided fractional derivative within the exponential decay kernel:

$$\begin{aligned}&\mathrm{D}_{b-}^{\alpha,\lambda} f(t)=\\&=\int_t^b \exp\left[-\lambda(\tau-t)\right]\left[{}_{\mathrm{LSC}}\mathrm{D}_{b-}^{\alpha} f(\tau)\right]\mathrm{d}\tau=\\&=\int_t^b \exp\left[-\lambda(\xi-\varsigma)\right]\left[\frac{1}{\Gamma(\kappa-\alpha)}\int_t^{b}\frac{f^{(\kappa)}(\tau)}{(\tau-\varsigma)^{\alpha-\kappa+1}}\mathrm{d}\tau\right]\mathrm{d}\xi\end{aligned} \tag{16}$$

In particular, when $0 \leq \alpha \leq 1$, the left-sided fractional derivative within the exponential decay kernel is defined:

$$\begin{aligned}&\mathrm{D}_{a+}^{\alpha,\lambda} f(t)=\\&=\int_a^t \exp\left[-\lambda(t-\tau)\right]\left[{}_{\mathrm{LSC}}\mathrm{D}_{a+}^{\alpha} f(\tau)\right]\mathrm{d}\tau=\\&=\int_a^t \exp\left[-\lambda(\varsigma-\xi)\right]\left[\frac{1}{\Gamma(1-\alpha)}\int_a^{\varsigma}\frac{f^{(1)}(\tau)}{(\varsigma-\tau)^{\alpha}}\mathrm{d}\tau\right]\mathrm{d}\xi\end{aligned} \tag{17}$$

and the right-sided fractional derivative within the exponential decay kernel:

$$\begin{aligned}&\mathrm{D}_{b-}^{\alpha,\lambda} f(t)=\\&=\int_t^b \exp\left[-\lambda(\tau-t)\right]\left[{}_{\mathrm{LSC}}\mathrm{D}_{b-}^{\alpha} f(\tau)\right]\mathrm{d}\tau=\\&=\int_t^b \exp\left[-\lambda(\xi-\varsigma)\right]\left[\frac{1}{\Gamma(1-\alpha)}\int_t^b \frac{f^{(1)}(\tau)}{(\tau-\varsigma)^{\alpha}}\mathrm{d}\tau\right]\mathrm{d}\xi\end{aligned} \tag{18}$$

As the inverse operators of the fractional derivatives, we define the corresponding fractional integrals:

The left-sided fractional integral of the function $f(t)$ is defined:

$$I_{a+}^{\alpha,\lambda} f(t)=\int_a^t (t-\tau)^{\alpha} E_{-1,\alpha}^{-1}\left[-\lambda(t-\tau)^{-1}\right] f(\tau)\mathrm{d}\tau \tag{19}$$

and the right-sided fractional integral of the function $f(t)$:

$$I_{+}^{\alpha,\lambda} f(t)=\int(\tau-t)\ \ E_{-1,}^{1}\left[-\lambda(\tau-t)\ \ \right] f(\tau)\ \ \tau \tag{20}$$

where $E_{\alpha,\upsilon}^{\phi}(\lambda t^{\alpha})$ is the Prabhakar function [1, 4, 16].

The properties of the fractional derivatives within the exponential decay kernel are presented:

– $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$ and $\lambda > 0$, then we have:

$$L\left\{\mathrm{D}_{0+}^{\alpha,\lambda} f(t)\right\}=\frac{s^{\alpha-\kappa}}{s+\lambda}\left[s^{\kappa} f(s)-\sum_{j=1}^{\kappa} s^{\kappa-j} f^{(j-1)}(0)\right] \tag{21}$$

– If $0 \le \alpha < 1$ and $\lambda > 0$, then we have:

$$L\left\{D_{0+}^{\alpha,\lambda} f(t)\right\}=\frac{s^{\alpha}}{s+\lambda} f(s)-s^{\alpha-1} f(0) \tag{22}$$

– If $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$ and $\lambda > 0$, then we have:

$$L\left\{I_{0+}^{\alpha,\lambda} f(t)\right\}=(s+\lambda)s^{-\alpha} f(s) \tag{23}$$

– If $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$ and $\lambda > 0$, then we have:

$$\mathrm{D}_{a+}^{\alpha,\lambda} I_{a+}^{\alpha,\lambda} f(t)=f(t) \tag{24}$$

– If $\kappa \le \alpha < \kappa + 1$, $\kappa \ge 0$, $\kappa \in \mathbb{N}_0$ and $-\infty < a < b < \infty$ and $\lambda > 0$, then we have:

$$I_{a+}^{\alpha,\lambda}\mathrm{D}_{a+}^{\alpha,\lambda} f(t)=f(t)-\sum_{j=0}^{\kappa-1} f^{(j)}(a)\frac{(t-a)^{j}}{\Gamma(1+j)} \tag{25}$$

and

$$I_{b-}^{\alpha,\lambda}\mathrm{D}_{b-}^{\alpha,\lambda} f(t)=f(t)-\sum_{j=0}^{\kappa-1}(-1)^{j} f^{(j)}(b)\frac{(b-t)^{j}}{\Gamma(1+j)} \tag{26}$$

A new anomalous diffusion model involving fractional derivative within the exponential decay kernel

In this section, we propose the new time-fractional anomalous diffusion model involving fractional derivative within the exponential decay kernel.

A fractional-derivative diffusion model within the exponential decay kernel is written:

$$\frac{\partial_{0+}^{\alpha,\lambda} f(u,t)}{\partial t^{\alpha}} = \kappa \frac{\partial^2 f(u,t)}{\partial u^2} \tag{27}$$

subjected to the initial and boundary conditions:

$$f(u,0) = 0 \tag{28}$$

$$f(0,t) = \delta(t) \tag{29}$$

$$f(u,t) \to 0 \quad \text{as} \quad u \to \infty, \quad t>0 \tag{30}$$

where

$$\frac{\partial_{0+}^{\alpha,\lambda} f(u,t)}{\partial t^{\alpha}} = \int_a^t \exp\left[-\lambda(\varsigma - \xi)\right]\left[\frac{1}{\Gamma(1-\alpha)}\int_a^{\varsigma} \frac{f^{(1)}(u,\tau)}{(\varsigma-\tau)^{\alpha}}\mathrm{d}\tau\right]\mathrm{d}\xi \tag{31}$$

and κ is the diffusive constant.

With the use of:

$$L\left\{\mathrm{D}_{0+}^{\alpha,\lambda} f(u,t)\right\} = \frac{s^{\alpha}}{s+\lambda} f(u,s) - s^{\alpha-1} f(u,0) \tag{32}$$

we obtain:

$$\frac{\mathrm{d}^2 f(u,s)}{du^2} - \frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda} f(u,s) = 0 \tag{33}$$

The general solution of eq. (27) is given:

$$f(u,s) = \gamma_1 \mathrm{e}^{-u\sqrt{\frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda}}} + \gamma_2 \mathrm{e}^{u\sqrt{\frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda}}} \tag{34}$$

which, by using the eq. (30), leads:

$$\gamma_2 = 0 \tag{35}$$

and

$$f(u,s) = \gamma_1 \mathrm{e}^{-u\sqrt{\frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda}}} \tag{36}$$

Making use of eq. (29), we rewrite eq. (36):

$$f(u,s) = \mathrm{e}^{-u\sqrt{\frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda}}} \tag{37}$$

which leads to:

$$f(u,s) = \mathrm{e}^{-u\sqrt{\frac{1}{\kappa}\frac{s^{\alpha}}{s+\lambda}}} = \sum_{j=0}^{\infty} \frac{(-u)^j}{\Gamma(j+1)} \kappa^{-j/2} \frac{s^{j\alpha/2}}{(s+\lambda)^{j/2}} \tag{38}$$

Taking the inverse Laplace of eq. (38), we give the fundamental solution of eq. (27) in the form of the Prabhakar function:

$$f(u,t) = \sum_{j=0}^{\infty} \frac{(-u)^j}{\Gamma(j+1)} \kappa^{-j/2} t^{-(j\alpha/2)-1} E_{-1,-(j\alpha/2)}^{j/2}\left(-\lambda t^{-1}\right) \tag{39}$$

and the corresponding plots for α = 0.3, α = 0.5, and α = 0.9 are depicted in figs. 1-3, respectively.

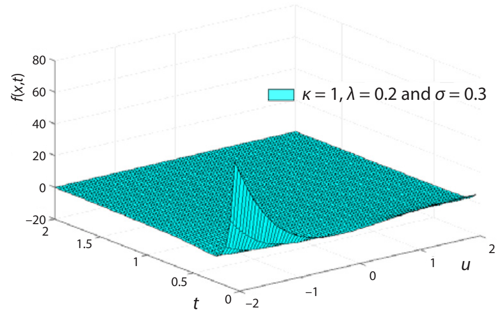

Figure 1. The solution of the fractional-derivative diffusion model within the exponential decay kernel when $\kappa = 1$, $\lambda = 0.2$, and $\alpha = 0.3$

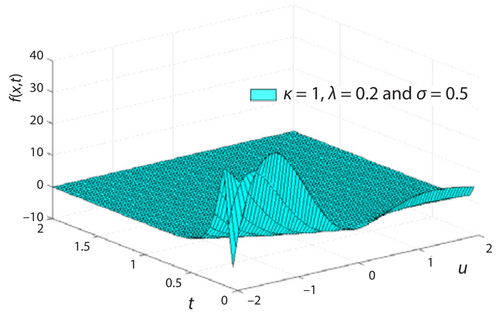

Figure 2. The solution of the fractional-derivative diffusion model within the exponential decay kernel when $\kappa = 1$, $\lambda = 0.2$, and $\alpha = 0.5$

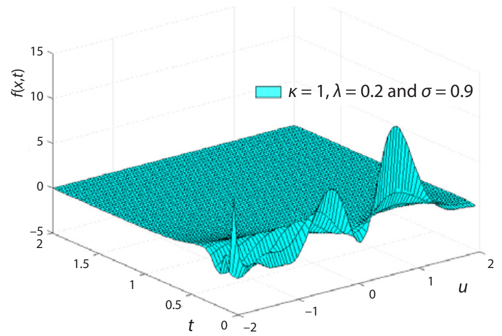

Figure 3. The solution of the fractional-derivative diffusion model within the exponential decay kernel when $\kappa = 1$, $\lambda = 0.2$, and $\alpha = 0.9$

Conclusion

In the present work, we proposed a new fractional derivative within the exponential decay kernel which is the mixed fractional derivative based on the Liouville-Sonine and Liouville-Sonine-Caputo fractional derivatives and the general derivatives. The Laplace transforms of the new fractional derivative within the exponential decay kernel and the corresponding integral operator are given. The new time-fractional anomalous diffusion model involving fractional derivative within the exponential decay kernel and the solutions with the different conditions are discussed in detail. The proposed fractional calculus operators are efficient and accurate in the description of the anomalous diffusion behaviors of the heat-conduction problem in the complex media.

Acknowledgment

This work was supported by the financial support of the 333 Project of Jiangsu Province, People's Republic of China (Grant No. BRA2018320), the Yue-Qi Scholar of the China University of Mining and Technology (Grant No. 102504180004) and the State Key Research Development Program of the People's Republic of China (Grant No. 2016YFC0600705).

Nomenclature

x – space co-ordinate, [m]

Greek symbols

α – fractional order, [–]

κ – diffusive constant, [m^2s^{-1}]

τ – time, [s]

References

[1] Kilbas, A. A., *et al*., *Theory and Applications of Fractional Differential Equations*, Elsevier Science Limited, Amsterdam, The Netherland, 2006
[2] Baleanu, D., *et al.*, *Fractional Calculus: Models and Numerical Methods*, World Scientific, Singapore, 2012
[3] Yang, X. J., *et al.*, *Local Fractional Integral Transforms and their Applications*, Academic Press, New York, USA, 2015
[4] Yang, X. J., *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, Boca Raton, Fla., USA, 2019
[5] Chechkin, A. V., *et al.*, Distributed Order Time Fractional Diffusion Equation, *Fractional Calculus and Applied Analysis*, *6* (2003), 3, pp. 259-280
[6] Li, X., *et al*., A Space-Time Spectral Method for the Time Fractional Diffusion Equation, *SIAM Journal on Numerical Analysis*, *47* (2009), 3, pp. 2108-2131
[7] Tejedor, V., *et al*., Anomalous Diffusion in Correlated Continuous Time Random Walks, *Journal of Physics A: Mathematical and Theoretical*, *43* (2010), 8, 082002
[8] Mainardi, F., *et al*., The Wright Functions as Solutions of the Time-Fractional Diffusion Equation, *Applied Mathematics and Computation*, *141* (2003), 1, pp. 51-62
[9] Yang, Q., *et al.*, Novel Numerical Methods for Solving the Time-Space Fractional Diffusion Equation in two Dimensions, *SIAM Journal on Scientific Computing*, *33* (2011), 3, pp. 1159-1180
[10] Liouville, J., Memoire sur quelques Questions de Gomtrie et de Mcanique, et sur un nouveau genre de Calcul pour rsoudre ces Questions, *Journal de l Ecole Polytechnique*, tome XIII, XXIe cahier, 1832, pp. 1-69
[11] Riemann, B., Versuch einer allgemeinen Auffassung der Integration und Differentiation, 14 Janvier 1847, Bernhard Riemanns Gesammelte Mathematische Werke, 1892, pp. 353-362
[12] Sonine, N., Sur la diffrentiation a indice quelconque, *Matematicheskii Sbornik*, *6* (1872), 1, pp. 1-38
[13] Caputo, M., Linear Models of Dissipation Whose Q is almost Frequency Independent II, *Geophysical Journal International*, *13* (1967), 5, pp. 529-539
[14] Caputo, M., *et al*., Fractional Derivative without Singular Kernel, *Progr. Fract. Differ*, *Appl*, *1* (2015), 2, pp. 1-13
[15] Yang, X. J., *et al.*, A New Fractional Derivative without Singular Kernel: Application the Modelling of the Steady Heat Flow, *Thermal Science*, *2* (2016), 20, pp. 753-756
[16] Prabhakar, T. R., A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, *Yokohama Mathematical Journal*, *1* (1971), 19, pp. 7-15

Paper submitted: September 12, 2018
Paper revised: January 18, 2019
Paper accepted: February 25, 2019