# A NEW METHOD SOLVING LOCAL FRACTIONAL DIFFERENTIAL EQUATIONS IN HEAT TRANSFER

by

***Yong-Ju YANG*** *

School of Mathematics and Statistics, Nanyang Normal University, Nanyang, China

Original scientific paper
https://doi.org/10.2298/TSCI180912237Y

*In this article, a new method, which is coupled by the variational iteration and reduced differential transform method, is proposed to solve local fractional differential equations. The advantage of the method is that the integral operation of variational iteration is transformed into the differential operation. One test examples is presented to demonstrate the reliability and efficiency of the proposed method.*

Key words: *variational iteration method, reduced differential transform method, local fractional differential equation*

## Introduction

In the last few decades, many types of fractional calculus have been introduced in [1]. However, most fractional derivatives are globally defined, so they are not suitable to reflect local geometric behaviors of a given function. Thence a few of local fractional derivatives [2-5] have been defined by modifying the definition of popular non-local fractional derivatives and making them local. Among these local fractional derivatives, one local fractional derivative [4] is developed to deal with non-differentiable functions defined on Cantor fractal space. The local fractional calculus [5] has been used to process many equations resulting from practical problems, for example, the non-linear local fractional FitzHugh-Nagumo and Newell-Whitehead equations [6], the fractal heat transfer in silk cocoon hierarchy [7], the electric circuit [8], the wave equation [9], *etc.* The list is obviously not complete. Many methods have been developed to find the analytic solutions of the local fractional differential equations, for example, the Yang-Fourier transforms method [10], the local fractional homotopy perturbation method [11], the fractional complex transform method [12], the local fractional Laplace series expansion method [13], the Sumudu transform and the variational iteration method [14], the Yang-Laplace method [15], the hybrid computational approach [16-18], *etc.* Among these methods, it is worth noting that the local fractional variational iteration method could be used to solve various types of local fractional differential equations. In this paper, by coupling the local differential transform method, we enrich the local fractional variational iteration method in the form of series, while maintaining the high accuracy of the method.

## Preliminaries

In this section, we introduce some definitions and basic results of the local fractional differential calculus, which shall be applied in this article.

* Author's e-mail: tomjohn1007@126.com

*Definition 1*. The $\alpha(0<\alpha\leq 1)$ order local fractional derivative of the function $\vartheta(x)$ at $x_0$ is defined [4, 5]:

$$\vartheta^{(\alpha)}(x_0)=\mathrm{D}_x^{(\alpha)}\vartheta(x_0)=\frac{d^\alpha\vartheta(x)}{dx^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Gamma(1+\alpha)\left[\vartheta(x)-\vartheta(x_0)\right]}{(x-x_0)^\alpha} \tag{1}$$

*Definition 2*. In the interval $[c, d]$, the local fractional integral of $\vartheta(t)$ of order $\alpha(0<\alpha\leq 1)$ is defined [4, 5]:

$${}_cI_d^{(\alpha)}\vartheta(t)=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}\vartheta(t_j)(\Delta t_j)^\alpha=\frac{1}{\Gamma(1+\alpha)}\int_c^d\vartheta(t)(dt)^\alpha \tag{2}$$

where $\Delta t_j=t_{j+1}-t_j$ and, $j=0,...,N-1$, $t_0=c$, $t_N=d$ is a partition of the interval $[c, d]$.

*Definition 3*. The 2-D local fractional reduced differential transform $\Theta(k, m)$ of the function $\vartheta(t, x)$ is defined by the following eq. [19]:

$$\Theta(k,m)=DT_{k,m}\left\{\vartheta(t,x)\right\}=\frac{1}{\Gamma(1+k\alpha)\Gamma(1+m\alpha)}\left[\frac{\partial^{k\alpha+m\alpha}}{\partial t^{k\alpha}\partial x^{m\alpha}}\vartheta(t,x)\right]_{x=0,t=0} \tag{3}$$

where

$$\vartheta(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\Theta(k,h)t^{k\alpha}x^{h\alpha} \tag{4}$$

## The process of the new method

In this section, in order to present the technique and the process of the method coupled variational iteration and reduced differential transform method, we give the following local fractional differential equation:

$$\frac{\partial^{k_o\alpha}}{\partial t^{k_0}}\vartheta(t,x)+L_\alpha\vartheta(t,x)+R_\alpha\vartheta(t,x)=\mu(t,x) \tag{5}$$

where $L_\alpha$ *and* $R_\alpha$ are linear and non-linear operator, respectively, which all have differential order less than $k_0(k_0\geq 1)$ with respect to variable, and $\mu(t, x)$ is the source term.

In light of the local fractional variational iteration formula, the correction function for eq. (5) is constructed [4]:

$$\vartheta_{n+1}(t,x)=\vartheta_n(t,x)+{}_0I_t^{(\alpha)}\left\{\frac{\kappa^\alpha(\tau,t)}{\Gamma(1+\alpha)}\left[\frac{\partial^{k_o\alpha}}{\partial\tau^{k_0}}\vartheta_n(\tau,x)+L_\alpha\tilde{\vartheta}_n(\tau,x)+R_\alpha\tilde{\vartheta}_n(\tau,x)-\mu(\tau,x)\right]\right\} \tag{6}$$

where $\kappa^\alpha(\tau, t)/\Gamma(1+\alpha)$ is a Lagrange multiplier.

If $\tilde{\vartheta}_n(\tau, x)$ is a restricted variation, *i. e.* $\delta^\alpha\tilde{\vartheta}_n(\tau, x)=0$, according to [4], we can derive:

$$\frac{\kappa^\alpha(\tau,t)}{\Gamma(1+\alpha)}=\frac{(\tau-t)^{(k_0-1)\alpha}}{\Gamma\left[1+(k_0-1)\alpha\right]} \tag{7}$$

From eqs. (6) and (7) we can construct the following iteration procedure for eq. (5):

$$\vartheta_{n+1}(t,x)=\vartheta_n(t,x)+{}_0I_t^{(\alpha)}\cdot$$

$$\cdot\left\{\frac{(\tau-t)^{(k_0-1)\alpha}}{\Gamma\left[1+(k_0-1)\alpha\right]}\left[\frac{\partial^{k_o\alpha}}{\partial\tau^{k_0}}\vartheta_n(\tau,x)+L_\alpha\vartheta_n(\tau,x)+R_\alpha\vartheta_n(\tau,x)-\mu(\tau,x)\right]\right\} \tag{8}$$

Taking the mixed partial derivative with respect to the variables $t$ and $x$ on both sides of eq. (8), we can obtain:

$$\frac{\partial^{k\alpha+h\alpha}}{\partial t^{k\alpha}\partial x^{h\alpha}}\vartheta_{n+1}(t,x)=\frac{\partial^{k\alpha+h\alpha}}{\partial t^{k\alpha}\partial x^{h\alpha}}\vartheta_n(t,x)+$$
$$+\frac{\partial^{h\alpha}}{\partial x^{h\alpha}}\left\{{}_0I_t^{(\alpha)}\left[\frac{(\tau-t)^{(k_0-1)\alpha}}{\Gamma\left[1+(k_0-1)\alpha\right]}\left(\frac{\partial^{k_o\alpha}}{\partial\tau^{k_0}}\vartheta_n+L_\alpha\vartheta_n+R_\alpha\vartheta_n-\mu(\tau,x)\right)\right]\right\},\ (0\le k\le k_0-1) \tag{9}$$

where $\vartheta_n = \vartheta_n(\tau, x)$, and

$$\frac{\partial^{k\alpha+h\alpha}}{\partial t^{k\alpha}\partial x^{h\alpha}}\vartheta_{n+1}(t,x)=-\frac{\partial^{k\alpha+h\alpha}}{\partial t^{k\alpha}\partial x^{h\alpha}}\left\{{}_0I_t^{(k_0\alpha)}\left[L_\alpha\vartheta_n(\tau,x)+R_\alpha\vartheta_n(\tau,x)-\mu(\tau,x)\right]\right\},(k\ge k_0) \tag{10}$$

According to eq. (9) and taking the 2-D local fractional reduced differential transform with respect to the variables $t$ and $x$ on both sides of eq. (8), we can obtain:

$$DT_{k,h}\left[\vartheta_{n+1}(t,x)\right]=DT_{k,h}\left[\vartheta_n(t,x)\right]=...=DT_{k,h}\left[\vartheta_0(t,x)\right],\ (0\le k\le k_0-1) \tag{11}$$

where $\vartheta_0(t, x)$ is an initial value.

With the use of eq. (10) and taking the 2-D local fractional reduced differential transform with respect to the variables $t$ and $x$ on both sides of eq. (8), we can obtain:

$$DT_{k,h}\left\{\vartheta_{n+1}(t,x)\right\}=-DT_{k,h}\left\{{}_0I_t^{(k_0\alpha)}\left[L_\alpha\vartheta_n(\tau,x)+R_\alpha\vartheta_n(\tau,x)-\mu(\tau,x)\right]\right\},\ (k\ge k_0) \tag{12}$$

According to eq. (3) and letting $\Theta(k, h) = DT_{k,h}\{\vartheta_n(\tau, x)\}$, $n = 0, 1, 2. 3...$, we can rewrite eqs. (11) and (12), respectively:

$$\Theta_{T_{n+1}}(k,h)=\Theta_{T_0}(k,h),\quad (0\le k\le k_0-1) \tag{13}$$

and

$$\Theta_{T_{n+1}}(k,h)=-\frac{\Gamma\left[1+(k-k_0)\alpha\right]}{\Gamma(1+k\alpha)}\Theta_{W_n}(k-k_0,h),\ (k\ge k_0) \tag{14}$$

where

$$\Theta_{W_n}(k-k_0,h)=DT_{k-k_0,h}\left\{W_n(t,x)\right\}=\frac{\frac{\partial^{(k-k_0+h)\alpha}}{\partial x^{(k-k_0)\alpha}\partial t^{h\alpha}}\left[W_n(t,x)\right]}{\Gamma\left[1+(k-k_0)\alpha\right]\Gamma(1+h\alpha)} \tag{15}$$

and where

$$W_n(t,x)=L_\alpha\vartheta_n(t,x)+R_\alpha\vartheta_n(t,x)-\mu(t,x) \tag{16}$$

Now, we are reminded of the following convergence conditions of the variational iteration method in [20]. If $\vartheta_{n+1}(t, x)$ satisfies:

$$\left\|\vartheta_{n+1}(t,x)-\vartheta_n(t,x)\right\|_\alpha\le\gamma^\alpha\left\|\vartheta_n(t,x)-\vartheta_{n\text{-}1}(t,x)\right\|_\alpha,\ 0<\gamma<1 \tag{17}$$

where $\gamma$ is a constant and $\|\bullet\|_\alpha$ is fractional operator norm, then $\lim_{n\to\infty}\vartheta_n(t, x)$ converges and the convergence function is an exact solution of the eq. (5).

In conclusion, if $\vartheta_{n+1}(t, x)$ satisfies eq. (17), the analytic solution of eq. (5) can be given by the following series form:

$$\vartheta(t,x)=\lim_{n\to\infty}\vartheta_{n+1}(t,x)=\lim_{n\to\infty}\left\{\sum_{h=0}^{\infty}\sum_{k=0}^{k_0-1}\Theta_{T_0}(k,m)t^{k\alpha}x^{h\alpha}+\sum_{h=0}^{\infty}\sum_{k=k_0}^{\infty}\Theta_{T_{n+1}}(k,h)t^{k\alpha}x^{h\alpha}\right\}$$

$$=\sum_{h=0}^{\infty}\sum_{k=0}^{k_0-1}\Theta_{T_0}(k,m)t^{k\alpha}x^{h\alpha}-\lim_{n\to\infty}\sum_{h=0}^{\infty}\sum_{k=k_0}^{\infty}\frac{\Gamma(1+(k-k_0)\alpha)}{\Gamma(1+k\alpha)}\Theta_{W_n}(k-k_0,h)x^{k\alpha}t^{h\alpha} \tag{18}$$

The method indeed provides an efficient tool to solve differential equation. Now, we consider the exact solution of the following linear differential equations:

$$\frac{\partial^{2\alpha}\vartheta(t,x)}{\partial t^{2\alpha}}+\rho_1\frac{\partial^{2\alpha}\vartheta(t,x)}{\partial x^{2\alpha}}+\rho_2\frac{\partial^{\alpha}\vartheta(t,x)}{\partial t^{\alpha}}+\rho_3\frac{\partial^{\alpha}\vartheta(t,x)}{\partial x^{\alpha}}+\rho_4\vartheta(t,x)=0 \tag{19}$$

where $\rho_i$, ($i = 1, 2, 3, 4$) are all constants. Obviously, eq. (19) is an essential special case of eq. (5).

By virtue of Eq. (8), the iteration algorithm is offered:

$$\vartheta_{n+1}(t,x)=\vartheta_n(t,x)+{}_0I_t^{\alpha}\left\{\frac{(\tau-t)^{(\alpha)}}{\Gamma(1+\alpha)}\left[L_{tt}^{(2\alpha)}\vartheta_n+\rho_1L_{xx}^{(2\alpha)}\vartheta_n+\rho_2L_t^{(\alpha)}\vartheta_n+\rho_3L_x^{(\alpha)}\vartheta_n+\rho_4\vartheta_n\right]\right\} \tag{20}$$

where $\vartheta_n=\vartheta_n(\tau, x)$.

Taking local fractional reduced differential transform with respect to the variables $t$ and $x$ on both sides of eq. (20), we successively obtain systems of equations:

$$\begin{cases}DT_{k,h}\left[\vartheta_{n+1}(t,x)\right]=-DT_{k-2,h}\left(\rho_1L_{xx}^{(2\alpha)}\vartheta_n+\rho_2L_t^{(\alpha)}\vartheta_n+\rho_3L_x^{(\alpha)}\vartheta_n+\rho_4\vartheta_n\right),\ (k\ge2)\\ DT_{k,h}\left[\vartheta_{n+1}(t,x)\right]=DT_{k,h}\left[\vartheta_0(t,x)\right],\ (k=0,1)\end{cases} \tag{21}$$

Employing eq. (10), we rewrite eq. (21) as the following eqs. (22) and (23):

$$\Theta_{T_{n+1}}(k,h)=-\rho_1\frac{(h+1)\alpha(h+2)\alpha\,\Theta_{T_n}(k-2,h+2)}{k\alpha(k-1)\alpha}-\rho_2\frac{(k+1)\alpha\,\Theta_{T_n}(k+1,h)}{k\alpha(k-1)\alpha}-$$

$$-\rho_3\frac{(h+1)\alpha\,\Theta_{T_n}(k,h+1)}{k\alpha(k-1)\alpha}-\rho_4\frac{\Theta_{T_n}(k-2,h)}{k\alpha(k-1)\alpha},\ (k\ge2) \tag{22}$$

and

$$\Theta_{T_{n+1}}(k,h)=\Theta_{T_0}(k,h),\ 0\le k\le1 \tag{23}$$

$$\vartheta(t,x)=\lim_{n\to\infty}\vartheta_{n+1}(t,x)=\sum_{h=0}^{\infty}\sum_{k=0}^{1}\Theta_{T_0}(k,h)t^{k\alpha}x^{h\alpha}-$$

$$-\lim_{n\to\infty}\sum_{h=0}^{\infty}\sum_{k=2}^{\infty}\left\{\begin{array}{l}\rho_1\dfrac{(h+1)\alpha(h+2)\alpha\,\Theta_{T_n}(k-2,h+2)}{k\alpha(k-1)\alpha}+\rho_2\dfrac{(k+1)\alpha\,\Theta_{T_n}(k+1,h)}{k\alpha(k-1)\alpha}\\ +\rho_3\dfrac{(h+1)\alpha\,\Theta_{T_n}(k,h+1)}{k\alpha(k-1)\alpha}+\rho_4\dfrac{\Theta_{T_n}(k-2,h)}{k\alpha(k-1)\alpha}\end{array}\right\}t^{k\alpha}x^{h\alpha} \tag{24}$$

## An illustrative example

To illustrate the techniques presented in the previous section, we give the following example to demonstrate the effectiveness of our proposed method.

Consider the following local fractional Laplace equation:

$$\frac{\partial^{2\alpha}\vartheta(t,x)}{\partial t^{2\alpha}}+\frac{\partial^{2\alpha}\vartheta(t,x)}{\partial x^{2\alpha}}=0 \tag{25}$$

subject to the initial condition:

$$\vartheta(0,x)=1+\cos_\alpha\left(x^\alpha\right) \tag{26}$$

According to eqs. (22), (23), and (25) the following iteration relation can be derived:

$$\Theta_{T_{n+1}}(k,h)=-\frac{(h+1)\alpha(h+2)\alpha\,\Theta_{T_n}(k-2,h+2)}{k\alpha(k-1)\alpha},\ (k\geq 2) \tag{27}$$

and

$$\Theta_{T_{n+1}}(k,h)=\Theta_{T_0}(k,h),(k=0,1) \tag{28}$$

By virtue of eq. (26), we take the initial value:

$$\vartheta_0(0,x)=1+\cos_\alpha\left(x^\alpha\right) \tag{29}$$

From eq. (29), we derive:

$$\Theta_{T_0}(0,h)=\begin{cases}2^\alpha, h=0\\ 0, h=2l+1, l\in N\\ (-1)^l\dfrac{1}{\Gamma(1+h\alpha)}, h=2l, l\in N, l\neq 0\end{cases} \tag{30}$$

and

$$\Theta_{T_0}(1,h)=0 \tag{31}$$

Using eqs. (27), (28), and (31), we can get:

$$\Theta_{T_{n+1}}(k,h)=0,\ k=2p+1,\ p\in N \tag{32}$$

Similarly, using eqs. (27), (28), and (30), we can get:

$$\Theta_{T_{n+1}}(0,h)=\begin{cases}2^\alpha, h=0\\ 0, h=2l+1, l\in N\\ (-1)^l\dfrac{1}{\Gamma(1+h\alpha)}, h=2l, l\in N, l\neq 0\end{cases} \tag{33}$$

$$\Theta_{T_{n+1}}(2,h)=\begin{cases}0, h=2l+1, l\in N\\ (-1)^l\dfrac{1}{\Gamma(1+2\alpha)\Gamma(1+h\alpha)}, h=2l, l\in N\end{cases} \tag{34}$$

$$\Theta_{T_{n+1}}(4,h)=\begin{cases}0, h=2l+1, l\in N\\ (-1)^l\dfrac{1}{\Gamma(1+4\alpha)\Gamma(1+h\alpha)}, h=2l, l\in N\end{cases} \tag{35}$$

Then, according to eqs. (32)-(35), we can induce:

$$\Theta_{T_{n+1}}(k,h)=\begin{cases} 2^{\alpha}, k=h=0 \\ 0, k=2p+1 \\ 0, k=2p, h=2l+1 \\ (-1)^{l}\dfrac{1}{\Gamma(1+k\alpha)\Gamma(1+h\alpha)}, k=2p, h=2l, p\neq 0 \end{cases} \tag{36}$$

where $p$ and $l \in N$.

From eq. (24), the exact solution of eq. (25) can be given:

$$\vartheta(t,x)=1+\frac{1}{2}\left[E_{\alpha}\left(t^{\alpha}\right)+E_{\alpha}\left(-t^{\alpha}\right)\right]\cos_{\alpha}\left(x^{\alpha}\right) \tag{37}$$

## Conclusion

In this task, we tried to couple the local fractional variational iteration with reduced differential transform method for solving local fractional differential equations. The example shows that our method can be easier to analyze the convergence solution or approximate solution.

## Acknowledgment

This work is supported by the foundation of the Nanyang Normal University (2018STP005).

## Nomenclature

$t$ – time co-ordinate, [s]
$x$ – space co-ordinate, [m]

*Greek symbols*

$\alpha$ – fractal order, [–]
$\vartheta(t, x)$ – temperature distribution, [K]

## References

[1] Yang, X. J., *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, New York, USA, 2019
[2] Kolwankar, K. M., *et al.,* Fractional Differentiability of Nowhere Differentiable Functions and Dimensions, *Chaos, 6* (1996), 4, pp. 505-513
[3] Babakhani, A., *et al.*, On Calculus of Local Fractional Derivatives. Journal Math. Anal. Appl., *270* (2002), 1, pp. 66-79
[4] Yang, X. J., *et al., Local Fractional Integral Transforms and Their Applications*, Academic Press, New York, USA, 2015
[5] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications,* World Science Publisher, New York, USA, 2012
[6] Yang, X. J., *et al.,* A New Computational Approach for Solving Non-Linear Local Fractional PDE, *Journal of Computational and Applied Mathematics*, *339* (2018), Jan., pp. 285-296
[7] He, J. H., *et al.*, Local Fractional Variational Iteration Method for Fractal Heat Transfer in Silk Cocoon Hierarchy, *Non-Linear Science Letters A*, 4 (2013), 1, pp. 15-20
[8] Yang, X. J., *et al.,* On a Fractal LC-Electric Circuit Modeled by Local Fractional Calculus, *Communications in Non-Linear Science and Numerical Simulation*, *47* (2017), Jun., pp. 200-206
[9] Yang, X. J., *et al.*, New Rheological Models within Local Fractional Derivative, *Romanian Reports in Physics*, *69* (2017), 3, pp. 1-8
[10] Hemeda, A. A., *et al.*, Local Fractional Analytical Methods for Solving Wave Equations with Local Fractional Derivative, Mathematical Methods in the Applied Sciences, *41* (2018), 6, pp. 2515-2529
[11] Yang, X. J., *et al.*, Local Fractional Homotopy Perturbation Method for Solving Fractal Partial Differential Equations Arising in Mathematical Physics, *Romanian Reports in Physics*, *67* (2015), 3, pp. 752-761

[12] Su, W. H., *et al.*, Fractional Complex Transform Method for Wave Equations on Cantor Sets within Local Fractional Differential Operator, *Advances in Difference Equations*, *2013* (2013), 1, pp. 97-105
[13] Yang, X. J., *et al.*, New Analytical Solutions for Klein-Gordon and Helmholtz Equations in Fractal Dimensional Space, *Proceedings of the Romanian Academy Series A, Mathematics Physics Technical Sciences Information Science*, 18 (2017), 3, pp. 231-238
[14] Goswami, P., Solutions of Fractional Differential Equations by Sumudu Transform and Variational Iteration Method, *Journal of Non-Linear Science and Applications*, *9* (2016), 4, pp. 1944-1951
[15] Usta, F., *et al.,* Yang-Laplace Transform Method Volterra and Abel's Integro-Differential Equations of Fractional Order, *International Journal of Non-Linear Analysis and Applications*, *9* (2018), 2, pp. 203-214
[16] Zhao, D, *et al.*, An Efficient Computational Technique for Local Fractional Heat Conduction Equations in Fractal Media, *Journal of Non-Linear Sciences and Applications, 10* (2017), pp. 1478-1486
[17] Singh, J., *et al.,* A Reliable Algorithm for Local Fractional Tricomi Equation Arising in Fractal Transonic Flow, *Entropy*, *18* (2016), 6, pp. 1-8
[18] Kumar, *et al.,* A Hybrid Computational Approach for Klein- Gordon Equations on Cantor Sets, *Non-Linear Dynamics, 87* (2017), 1, pp. 511-517
[19] Jafari, H., *et al.,* Reduced Differential Transform Method for Partial Differential Equations within Local Fractional Derivative Operators, *Advances in Mechanical Engineering*, *8* (2016), 4, pp. 1-8
[20] Odibat, Z. M., A Study on the Convergence of Variational Iteration Method, *Mathematical and Computer Modelling, 51* (2010), 9, pp. 1181-1192

Paper submitted: September 12, 2018
Paper revised: November 1, 2018
Paper accepted: December 21, 2018