A NEW GENERAL FRACTIONAL-ORDER DERIVATIVE WITH RABOTNOV FRACTIONAL-EXPONENTIAL KERNEL

by

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In this article, a general fractional-order derivative of the Riemann-Liouville type with the non-singular kernel involving the Rabotnov fractional-exponential function is addressed for the first time. A new general fractional-order derivative model for the anomalous diffusion is discussed in detail. The general fractional-order derivative operator formula is as a novel and mathematical approach proposed to give the generalized presentation of the physical models in complex phenomena with power law.

Key words: *anomalous diffusion, general fractional-order derivative, power law Rabotnov fractional-exponential function, non-singular kernel*

Introduction

General fractional calculus (GFC) [1-4], as a general version of FC acting on the singular (power-law) kernel, *e. g.*, Liouville [5], Riemann [6], Weyl [7], Sonine [8], Caputo [9] and others (see [1]), has been successfully applied to describe some physical processes in complex phenomena. The general fractional-order derivatives (FD) and general fractional-order integrals (FI) with the non-singular kernels of the functions, such as the exponential function [10], Miller-Ross function [11], Lorenzo-Hartley function [12], Gorenflo-Mainardi function [13], Bessel function [14], Mittag-Leffler function [15], Wiman function [16], Prabhakar function [17], sinc function [18], and others [19].

In 1948, the fractional exponential function, also called the Rabotnov fractional exponential (RFE) function [1], was proposed by Rabotnov [20] and developed to model the internal friction given in [21]. The general FD in the sense of the Liouville-Caputo type with the non-singular kernel of the RFE function was reported in [22]. However, the general FD in sense of the Riemann-Liouville type with the non-singular kernel of the RFE function have not been considered to the best of our knowledge.

By the motivation of the tasks involving the physical phenomena with power-law and complex behaviors following the RFE function, the target of the paper is to derive the general FD of the Riemann-Liouville type with the non-singular kernel involving the RFE function and their properties, and to present a general FD model for the anomalous diffusion.

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A new GFC of Riemann-Liouville type with the RFE kernel

Suppose that $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$, and $\mathbb{N}_0$ are the sets of complex numbers, real numbers, non-negative real numbers, positive integers and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, respectively. Let $L(a,b)$ be the set of those Lebesgue measurable functions on a finite interval (a,b) $(-\infty \le a \le b \le +\infty)$ (for more details, see [1, 14]). Suppose that $AC(a,b)$ $(-\infty \le a \le b \le +\infty)$ and $AC^{\kappa}(a,b)$ $(-\infty \le a \le b \le +\infty)$ are the Kolmogorov-Fomin condition [1, 23], and the Samko-Kilbas-Marichev condition [1, 14], respectively.

General FI with the RFE kernel

The general FI with the RFE kernel on a finite interval (a,b) $(-\infty \le a \le b \le +\infty)$ is given:

$$\left({}_a\mathbb{I}_\tau^{(\alpha)}\Pi\right)(\tau) = {}_a\mathbb{I}_\tau^{(\alpha)}\Pi(\tau) = \int_a^\tau \mathrm{M}_\alpha\left[-\gamma(\tau-t)^\alpha\right]\Pi(t)\mathrm{d}t \tag{1}$$

where $\Pi \in L(a,b)$, $\gamma \in \mathbb{R}_0^+$, and the RFE function is defined as [1, 20-22]:

$$\mathrm{M}_\alpha(-\gamma t^\alpha) = \sum_{\rho=0}^{\infty} \frac{(-\gamma)^\rho t^{(\rho+1)(\alpha+1)-1}}{\Gamma\left[(\rho+1)(\alpha+1)\right]} \tag{2}$$

with $\rho \in \mathbb{N}_0$.

From eq. (1) we have [22]:

$$\left({}_0\mathbb{I}_\tau^{(\alpha)}\Pi\right)(\tau) = {}_0\mathbb{I}_\tau^{(\alpha)}\Pi(\tau) = \int_0^\tau M_\alpha\left[-\gamma(\tau-t)^\alpha\right]\Pi(t)\mathrm{d}t \tag{3}$$

where $a = 0$, $\Pi \in L(a,b)$ and $\gamma \in \mathbb{R}_0^+$, and from [22], we have:

$$\left(\mathbb{I}_+^{(\alpha)}\Pi\right)(\tau) = \mathbb{I}_+^{(\alpha)}\Pi(\tau) = \int_{-\infty}^\tau M_\alpha\left[-\gamma(\tau-t)^\alpha\right]\Pi(t)\mathrm{d}t \tag{4}$$

where $\Pi \in L(-\infty,b)$ and $\gamma \in \mathbb{R}_0^+$, and from [22] we have:

$$\left({}_0\mathbb{I}_{+\infty}^{(\alpha)}\Pi\right)(\tau) = {}_0\mathbb{I}_{+\infty}^{(\alpha)}\Pi(\tau) = \int_0^{+\infty} M_\alpha\left[-\gamma(\tau-t)^\alpha\right]\Pi(t)\mathrm{d}t \tag{5}$$

where $\Pi \in L(0,+\infty)$ and $\gamma \in \mathbb{R}_0^+$.

General FD of the Liouville-Caputo type with the RFE kernel

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on a finite interval (a,b) is given as [22]:

$$\left({}^{\mathrm{LC}}_{\ a}\mathbb{D}_\tau^{(\alpha)}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{\ a}\mathbb{D}_\tau^{(\alpha)}\Pi(\tau) = {}_a\mathbb{I}_\tau^{(\alpha)}\left[\Pi^{(1)}(\tau)\right] = \int_a^\tau M_\alpha\left[-\gamma(\tau-t)^\alpha\right]\Pi^{(1)}(t)\mathrm{d}t \tag{6}$$

and the right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on a finite interval (a,b) as [22]:

$$\left({}^{\mathrm{LC}}_{\tau}\mathbb{D}^{(\alpha)}_{b}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{\tau}\mathbb{D}^{(\alpha)}_{b}\Pi(\tau) = {}_{\tau}\mathbb{I}^{(\alpha)}_{b}\left[\Pi^{(1)}(\tau)\right] = -\int_{\tau}^{b} M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi^{(1)}(t)\mathrm{d}t \tag{7}$$

where $\Pi \in AC(a,b)$ and $\gamma \in \mathbb{R}_0^+$.

For $\alpha = 1$ we have the same results as in [1, 16].

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on a finite interval (a,b) is given as [22]:

$$\left({}^{\mathrm{LC}}_{a}\mathbb{D}^{(\alpha,n)}_{\tau}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{a}\mathbb{D}^{(\alpha,n)}_{\tau}\Pi(\tau) = {}_{a}\mathbb{I}^{(\alpha)}_{\tau}\left[\Pi^{(n)}(\tau)\right] = \int_{a}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi^{(n)}(t)\mathrm{d}t \tag{8}$$

and right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on a finite interval (a,b) is given as [22]:

$$\left({}^{\mathrm{LC}}_{\tau}\mathbb{D}^{(\alpha,n)}_{b}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{\tau}\mathbb{D}^{(\alpha,n)}_{b}\Pi(\tau) = {}_{\tau}\mathbb{I}^{(\alpha)}_{b}\left[\Pi^{(n)}(\tau)\right] = (-1)^{n}\int_{\tau}^{b} M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi^{(n)}(t)\mathrm{d}t \tag{9}$$

where $\Pi \in AC^{n}(a,b)$, $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}_0^+$.

For $a = 0$ we have from eqs. (6) and (9) that:

$$\left({}^{\mathrm{LC}}_{0}\mathbb{D}^{(\alpha)}_{\tau}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{0}\mathbb{D}^{(\alpha)}_{\tau}\Pi(\tau) = {}_{0}\mathbb{I}^{(\alpha)}_{\tau}\left[\Pi^{(1)}(\tau)\right] = \int_{0}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi^{(1)}(t)\mathrm{d}t \tag{10}$$

and

$$\left({}^{\mathrm{LC}}_{0}\mathbb{D}^{(\alpha,n)}_{\tau}\Pi\right)(\tau) = {}^{\mathrm{LC}}_{0}\mathbb{D}^{(\alpha,n)}_{\tau}\Pi(\tau) = {}_{0}\mathbb{I}^{(\alpha)}_{\tau}\left[\Pi^{(n)}(\tau)\right] = \int_{0}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi^{(n)}(t)\mathrm{d}t \tag{11}$$

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as [22]:

$$\left({}^{\mathrm{LC}}\mathbb{D}^{(\alpha)}_{+}\Pi\right)(\tau) = {}^{\mathrm{LC}}\mathbb{D}^{(\alpha)}_{+}\Pi(\tau) = \mathbb{I}^{(\alpha)}_{+}\left[\Pi^{(1)}(\tau)\right] = \int_{-\infty}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi^{(1)}(t)\mathrm{d}t \tag{12}$$

and the right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as [22]:

$$\left({}^{\mathrm{LC}}\mathbb{D}^{(\alpha)}_{-}\Pi\right)(\tau) = {}^{\mathrm{LC}}\mathbb{D}^{(\alpha)}_{-}\Pi(\tau) = \mathbb{I}^{(\alpha)}_{-}\left[\Pi^{(1)}(\tau)\right] = -\int_{\tau}^{+\infty} M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi^{(1)}(t)\mathrm{d}t \tag{13}$$

where $\Pi \in AC(-\infty,+\infty)$ and $\gamma \in \mathbb{R}_0^+$.

The left-sided general FD of the Liouville-Caputo type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as [22]:

$$\left({}^{\mathrm{LC}}\mathbb{D}^{(\alpha,n)}_{+}\Pi\right)(\tau) = {}^{\mathrm{LC}}\mathbb{D}^{(\alpha,n)}_{+}\Pi(\tau) = \mathbb{I}^{(\alpha)}_{+}\left[\Pi^{(n)}(t)\right] = \int_{-\infty}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi^{(n)}(t)\mathrm{d}t \tag{14}$$

and right-sided general FD of the Liouville-Caputo type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ is given as [22]:

$$\left({}^{\mathrm{LC}}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{\mathrm{LC}}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\mathbb{I}_{-}^{(\alpha)}\left[\Pi^{(n)}(\tau)\right]=(-1)^{n}\int_{\tau}^{+\infty}M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi^{(n)}(t)\mathrm{d}t \quad (15)$$

where $\Pi\in AC^{n}(-\infty,+\infty)$, $n\in\mathbb{N}$ and $\gamma\in\mathbb{R}_{0}^{+}$.

General FD of the Riemann-Liouville type with the RFE kernel

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on a finite interval (a,b) is defined as:

$$\left({}_{a}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{a}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[{}_{a}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{a}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (16)$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on a finite interval (a,b) as:

$$\left({}_{\tau}^{\mathrm{RL}}\mathbb{D}_{b}^{(\alpha)}\Pi\right)(\tau)={}_{\tau}^{\mathrm{RL}}\mathbb{D}_{b}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[{}_{\tau}\mathbb{I}_{b}^{(\alpha)}\Pi(\tau)\right]=-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\tau}^{b}M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (17)$$

where $\Pi\in L(a,b)$ and $\gamma\in\mathbb{R}_{0}^{+}$.

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on a finite interval (a,b) is defined as:

$$\left({}_{a}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(t)={}_{a}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)=\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\left[{}_{a}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\int_{a}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (18)$$

and the right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on a finite interval (a,b) as:

$$\left({}_{\tau}^{\mathrm{RL}}\mathbb{D}_{b}^{(\alpha,n)}\Pi\right)(\tau)={}_{\tau}^{\mathrm{RL}}\mathbb{D}_{b}^{(\alpha,n)}\Pi(\tau)=\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\left[{}_{\tau}\mathbb{I}_{b}^{(\alpha)}\Pi(\tau)\right]=(-1)^{n}\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\int_{\tau}^{b}M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (19)$$

where $\Pi\in L(a,b)$, $n\in\mathbb{N}$ and $\gamma\in\mathbb{R}_{0}^{+}$.

For $a=0$ we have from eqs. (16) and (19) that:

$$\left({}_{0}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha)}\Pi\right)(\tau)={}_{0}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[{}_{0}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{0}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (20)$$

and

$$\left({}_{0}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi\right)(\tau)={}_{0}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)=\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\left[{}_{0}\mathbb{I}_{\tau}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}^{n}}{\mathrm{d}\tau^{n}}\int_{0}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (21)$$

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is defined as:

$$\left({}^{\mathrm{RL}}\mathbb{D}_{+}^{(\alpha)}\Pi\right)(\tau)={}^{\mathrm{RL}}\mathbb{D}_{+}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathbb{I}_{+}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-\infty}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \quad (22)$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as:

$$\left({}^{\mathrm{RL}}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{\mathrm{RL}}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\mathbb{I}_{-}^{(\alpha)}\Pi(\tau)\right]=-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\tau}^{+\infty}M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t \tag{23}$$

where $\Pi\in L(-\infty,+\infty)$ and $\gamma\in\mathbb{R}_0^+$.

The left-sided general FD of the Riemann-Liouville type without the singular kernel of the RFE function on the real axis $\mathbb{R}$ is defined as:

$$\left({}^{\mathrm{RL}}\mathbb{D}_{+}^{(\alpha,n)}\Pi\right)(\tau)={}^{\mathrm{RL}}\mathbb{D}_{+}^{(\alpha,n)}\Pi(\tau)=\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\left[\mathbb{I}_{+}^{(\alpha)}\Pi(\tau)\right]=\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\int_{-\infty}^{\tau}M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t \tag{24}$$

and right-sided general FD of the Riemann-Liouville type with the non-singular kernel of the RFE function on the real axis $\mathbb{R}$ as:

$$\left({}^{\mathrm{RL}}\mathbb{D}_{-}^{(\alpha)}\Pi\right)(\tau)={}^{\mathrm{RL}}\mathbb{D}_{-}^{(\alpha)}\Pi(\tau)=\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\left[\mathbb{I}_{-}^{(\alpha)}\Pi(\tau)\right]=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}\tau^n}\int_{\tau}^{+\infty}M_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t \tag{25}$$

where $\Pi\in L(-\infty,+\infty)$, $n\in\mathbb{N}$ and $\gamma\in\mathbb{R}_0^+$.

For $\Pi(\tau)\big|_{\tau=0}=\Pi(0)$ there exists:

$${}_{a}^{\mathrm{LC}}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)={}_{a}^{\mathrm{RL}}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)-M_{\alpha}(-\gamma\tau^{\alpha})\Pi(0) \tag{26}$$

General FI via the Prabhakar function

The left-sided general FI of $\Pi(\tau)$ is given as [22]:

$${}_{a}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)=\int_{a}^{\tau}\Xi_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t=\int_{a}^{\tau}(\tau-t)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left[-\gamma(\tau-t)^{\alpha+1}\right]\Pi(t)\mathrm{d}t \tag{27}$$

and the right-sided general FI of $\Pi(\tau)$ as:

$${}_{\tau}\mathbb{I}_{b}^{(\alpha,n)}\Pi(\tau)=-\int_{\tau}^{b}\Xi_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t=-\int_{\tau}^{b}(t-\tau)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left[-\gamma(t-\tau)^{\alpha+1}\right]\Pi(t)\mathrm{d}t \tag{28}$$

where $\Pi\in L(a,b)$, $n\in\mathbb{N}$, $\gamma\in\mathbb{R}_0^+$, and $\Xi_{\alpha}(-\gamma\tau^{\alpha})=\tau^{n-(\alpha+2)}H_{\alpha+1,n-(\alpha+1)}^{-1}(-\gamma\tau^{\alpha+1})$ with the Prabhakar function, given as [1, 24]:

$$H_{\alpha,\beta}^{\gamma}(\tau)=\sum_{\rho=0}^{\infty}\frac{\Gamma(\gamma+\rho)}{\Gamma(\rho\alpha+\beta)\Gamma(\gamma)}\frac{\tau^{\rho}}{\Gamma(\rho+1)}$$

The left-sided general FI of $\Pi(\tau)$ is given as:

$$\mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)=\int_{-\infty}^{\tau}\Xi_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\Pi(t)\mathrm{d}t=\int_{-\infty}^{\tau}(\tau-t)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left[-\gamma(\tau-t)^{\alpha+1}\right]\Pi(t)\mathrm{d}t \tag{29}$$

and the right-sided general FI of $\Pi(\tau)$ as:

$$\mathbb{I}_{-}^{(\alpha,n)}\Pi(\tau)=-\int_{\tau}^{+\infty}\Xi_{\alpha}\left[-\gamma(t-\tau)^{\alpha}\right]\Pi(t)\mathrm{d}t=\int_{\tau}^{-\infty}(t-\tau)^{n-(\alpha+2)}E_{\alpha+1,n-(\alpha+1)}^{-1}\left[-\gamma(t-\tau)^{\alpha+1}\right]\Pi(t)\mathrm{d}t \tag{30}$$

where $\Pi\in L(-\infty,+\infty)$, $n\in\mathbb{N}$, and $\gamma\in\mathbb{R}_0^+$.

The properties for the general FD and FI are:
(I) Let $\Pi \in L(a,b)$ and $n \in \mathbb{N}$. Then ${}^{\mathrm{RL}}_{0}\mathbb{D}_{\tau}^{(\alpha,n)}({}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)) = \Pi(\tau)$,
(II) Let $\Pi \in (-\infty,+\infty)$ and $n \in \mathbb{N}$. Then ${}^{\mathrm{RL}}\mathbb{D}_{+}^{(\alpha,n)}(\mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)) = \Pi(\tau)$,
(III) Let $\Pi \in AC^{n}(a,b)$ and $n \in \mathbb{N}$. Then ${}^{\mathrm{LC}}_{0}\mathbb{D}_{\tau}^{(\alpha,n)}({}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(\tau)) = \Pi(\tau)$,
(IV) Let $\Pi \in (-\infty,+\infty)$ and $n \in \mathbb{N}$. Then ${}^{\mathrm{LC}}\mathbb{D}_{+}^{(\alpha,n)}(\mathbb{I}_{+}^{(\alpha,n)}\Pi(\tau)) = \Pi(\alpha)$.

The Laplace transforms of the general FD are:

$$G\left[{}^{\mathrm{RL}}_{0}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)\right] = s^{-\alpha}\left[1+\lambda s^{-(\alpha+1)}\right]^{-1}\Pi(s) - {}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\Pi(0) \tag{31}$$

and

$$G\left[{}^{\mathrm{RL}}_{0}\mathbb{D}_{\tau}^{(\alpha,n)}\Pi(\tau)\right] = s^{n-(\alpha+1)}\left[1+\lambda s^{-(\alpha+1)}\right]^{-1}\Pi(s) - \sum_{\eta=0}^{n-1} s^{n-\eta-1}\left\{\frac{\mathrm{d}^{\eta}}{\mathrm{d}\tau^{\eta}}\left[{}_{0}\mathbb{I}_{\tau}^{(\alpha,n)}\Pi(0)\right]\right\} \tag{32}$$

where the Laplace transform of $g(\tau)$ is [1]:

$$G\left[g(\tau)\right] = g(s) = \int_{0}^{\infty} \mathrm{e}^{-s\tau} g(\tau)\mathrm{d}\tau \tag{33}$$

with $s \in \mathbb{C}$.

For ${}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\Pi(0) = 0$ we have from eq. (31) that:

$$G\left[{}^{\mathrm{RL}}_{0}\mathbb{D}_{\tau}^{(\alpha)}\Pi(\tau)\right] = s^{-\alpha}\left[1+\lambda s^{-(\alpha+1)}\right]^{-1}\Pi(s) \tag{34}$$

A general FD diffusion model with the RFE kernel

We now consider the anomalous diffusion model containing the general FD of the Riemann-Liouville type with the RFE kernel:

$${}^{\mathrm{RL}}_{0}\partial_{\tau}^{(\alpha)}\psi(x,\tau) = \xi\frac{\partial^{2}\psi(x,\tau)}{\partial x^{2}} \tag{36}$$

with the initial condition ${}_{0}\mathbb{I}_{\tau}^{(\alpha,1)}\psi(x,0) = 0$ and the boundary conditions: $\psi(0,\tau) = 1$, $\psi(x,\tau) \to 0,\ x \to \infty,\ \tau > 0$, where ξ is the diffusivity constant, and

$${}^{\mathrm{RL}}_{0}\partial_{\tau}^{(\alpha)}\psi(x,\tau) = \frac{\partial}{\partial\tau}\int_{0}^{\tau} M_{\alpha}\left[-\gamma(\tau-t)^{\alpha}\right]\psi(x,\tau)\mathrm{d}t \tag{37}$$

With the use of the Laplace transform of eq. (36) with respect to the variable τ, we can get:

$$\frac{\mathrm{d}^{2}\psi(x,s)}{\mathrm{d}x^{2}} = \frac{s^{-\alpha}\left[1+\lambda s^{-(\alpha+1)}\right]^{-1}}{\xi}\psi(x,s) \tag{38}$$

which, due to the boundary conditions, this implies that:

$$\psi(x,s) = \mathrm{e}^{-x\sqrt{\frac{s^{-\alpha}\left[1+\lambda s^{-(\alpha+1)}\right]^{-1}}{\xi}}} = \sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)} s^{-\upsilon\alpha}\left[1+\lambda s^{-(\alpha+1)}\right]^{-\upsilon} \tag{39}$$

The general solution for eq. (36) can be represented as:

$$\psi(x,t)=\sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)}\left(\sum_{\rho=0}^{\infty}\frac{\Gamma(\upsilon+\rho)}{\Gamma(\rho\alpha+\upsilon\alpha)\Gamma(\upsilon)}\frac{(-\lambda)^{\rho}\tau^{(\alpha+1)\rho}}{\Gamma(\rho+1)}\right)=$$

$$=\sum_{\upsilon=0}^{\infty}\frac{\left(-\frac{x}{\sqrt{\xi}}\right)^{\upsilon}}{\Gamma(1+\upsilon)}\tau^{\upsilon\alpha-1}H_{\alpha+1,\upsilon\alpha}^{\upsilon}(-\lambda\tau^{\alpha+1}) \tag{40}$$

Conclusion

In the present work, we proposed the general FD of the Riemann-Liouville type with the non-singular kernel involving the RFE function. With the aid of the presented Laplace transforms, the general FD model for the anomalous diffusion with the solutions containing the Prabhakar function was investigated in detail. The formula of the general FD of the Riemann-Liouville type can be given to explore the mathematical models in physics and engineering practice.

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Nomenclature

- x – space co-ordinate, [m]
- α – fractional order, [–]
- ξ – diffusivity constant, [m^{2}s^{-1}]
- τ – time, [s]

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