THE NUMERICAL SOLUTION OF THE TIME-FRACTIONAL NON-LINEAR KLEIN-GORDON EQUATION VIA SPECTRAL COLLOCATION METHOD

by

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In this paper, we consider the numerical solution of the time-fractional non-linear Klein-Gordon equation. We propose a spectral collocation method in both temporal and spatial discretizations with a spectral expansion of Jacobi interpolation polynomial for this equation. A rigorous error analysis is provided for the spectral methods to show both the errors of approximate solutions and the errors of approximate derivatives of the solutions decaying exponentially in infinity-norm and weighted L^2-norm. Numerical tests are carried out to confirm the theoretical results.

Key words: *caputo derivative, non-linear, time-fractional Klein-Gordon equation, spectral collocation method*

Introduction

Fractional differential equations are the generalization of the traditional differential equations, and they play more and more important roles in the fields of fluid mechanics, material mechanics, biology, plasma physics and finance, and receive more and more attention [1]. The fractional diffusion equations are always used in describing the abnormal Klein-Gordon phenomenon [2-5] of the liquid in medium. The general form of the time fractional Klein-Gordon equation can be written:

$$\mathrm{D}_t^\gamma u(x,t)-\frac{\partial^2 u(x,t)}{\partial x^2}+\xi u(x,t)+\eta F\left[u(x,t)\right]=f(x,t),\ 1\le\gamma\le 2,\ a<x<b,\ 0<t<T \tag{1}$$

with initial conditions:

$$u(x,0)=\phi_1(x),\ u_t(x,0)=\phi_2(x),\ a<x<b \tag{2}$$

and boundary conditions:

$$u(a,t)=\psi_1(t),\ u(b,t)=\psi_2(t),\ 0<t<T \tag{3}$$

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Recently, we provided the Legendre-collocation methods and convergence analysis for the non-linear Volterra type integral equations in [6-8]. The main contribution of this work is constructing the Jacobi spectral collocation approximation in both space and time to the non-linear time-fractional Klein-Gordon equation and an analysis of the convergence of the proposed method.

The Jacobi collocation method

Applying the Riemann-Liouville integral of order γ at eq. (1), it yields the following integral equations:

$$u(x,t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{-\mu}\left\{\tilde{L}\left[t,\tau,u(x,\tau)\right]+\tilde{F}\left[t,\tau,u(x,\tau)\right]\right\}\mathrm{d}\tau+\tilde{f}(x,t)+\phi_1(x)+\phi_2(x)t \tag{4}$$

$$\tilde{L}\left[t,\tau,u(x,\tau)\right]=(t-\tau)\left[\frac{\partial^2 u(x,\tau)}{\partial x^2}-\xi u(x,\tau)\right],\tilde{F}\left[t,\tau,u(x,\tau)\right]=-(t-\tau)\eta F\left[u(x,\tau)\right]$$

$$\tilde{f}(x,t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1}f(x,\tau)\mathrm{d}\tau,\ \mu=2-\gamma\in(0,1)$$

In order to make use of orthogonal polynomials, we make the following transformation so that the variables in the standard interval [–1, 1]:

$$x=\frac{b-a}{2}\bar{x}+\frac{b+a}{2},\ t=\frac{T}{2}(1+\bar{t}),\ \tau=\frac{T}{2}(1+s)$$

$$\bar{x}=\frac{2x-a-b}{b-a},\ \bar{t}=\frac{2t}{T}-1,\ s=\frac{2\tau}{T}-1,\ \bar{x}\in[-1,1]\ \bar{t}\in[-1,1]\ s\in[-1,\bar{t}]$$

the eq. (4) can be written in the following form:

$$\bar{u}(\bar{x},\bar{t})=\frac{1}{\Gamma(\gamma)}\left(\frac{T}{2}\right)^{\gamma}\int_{-1}^{\bar{t}}(\bar{t}-s)^{-\mu}\left\{\hat{L}\left[\bar{t},s,\bar{u}(\bar{x},s)\right]+\hat{F}\left[\bar{t},s,\bar{u}(\bar{x},s)\right]\right\}\mathrm{d}s+\bar{f}(\bar{x},\bar{t})+\bar{\phi}(\bar{x},\bar{t}) \tag{5}$$

with boundary conditions:

$$\bar{u}(-1,\bar{t})=\bar{\psi}_1(\bar{t}),\quad \bar{u}(1,\bar{t})=\bar{\psi}_2(\bar{t}),\quad -1<\bar{t}<1 \tag{6}$$

$$\bar{u}(\bar{x},\bar{t})=u\left[\frac{b-a}{2}\bar{x}+\frac{a+b}{2},\frac{T}{2}(1+\bar{t})\right],\ \bar{f}(\bar{x},\bar{t})=\tilde{f}\left[\frac{b-a}{2}\bar{x}+\frac{a+b}{2},\frac{T}{2}(1+\bar{t})\right]$$

$$\hat{\mathcal{L}}\left[\bar{t},s,\bar{u}(\bar{x},s)\right]=(\bar{t}-s)\left[\left(\frac{2}{b-a}\right)^2\partial_{\bar{x}}^2\bar{u}(\bar{x},s)-\xi\bar{u}(\bar{x},s)\right],\ \hat{\mathcal{F}}\left[\bar{t},s,\bar{u}(\bar{x},s)\right]=(\bar{t}-s)\bar{\mathcal{F}}\left[\bar{u}(\bar{x},s)\right]$$

$$\bar{\phi}(\bar{x},\bar{t})=\phi_1\left(\frac{b-a}{2}\bar{x}+\frac{a+b}{2}\right)+\phi_2\left(\frac{b-a}{2}\bar{x}+\frac{a+b}{2}\right)\frac{T}{2}(1+\bar{t})$$

For the collocation methods in time, eq. (5) holds at the Jacobi collocation points $\{t_j\}_{j=1}^{M}$ with the Jacobi weight functions $\omega^{\alpha_t,\beta_t}(t)=(1-t)^{\alpha_t}_{\beta_t}(1+t)^{\alpha_t}_{\beta_t}$ on [–1, 1]:

$$\bar{u}(\bar{x},\bar{t}_j)=\frac{1}{\Gamma(\gamma)}\left(\frac{T}{2}\right)^{\gamma}\int_{-1}^{\bar{t}_j}(\bar{t}_j-s)^{-\mu}\{\hat{\mathcal{L}}[\bar{t}_j,s,\bar{u}(\bar{x},s)]+\hat{\mathcal{F}}[\bar{t}_j,s,\bar{u}(\bar{x},s)]\}\mathrm{d}s+\bar{f}(\bar{x},\bar{t}_j)+\bar{\phi}(\bar{x},\bar{t}_j) \tag{7}$$

We make the linear transformation:

$$s=s_j(\theta)=\frac{1+\bar{t}_j}{2}\theta+\frac{\bar{t}_j-1}{2},\ \theta\in[-1,1],\ \rho_j=\frac{1}{\Gamma(\gamma)}\left(\frac{T}{2}\right)^{\gamma}\left(\frac{1+\bar{t}_j}{2}\right)^{\gamma-1},\ \bar{\rho}_j=\frac{\bar{t}_j+1}{2}\rho_j \tag{8}$$

Thus, we transfer the integral interval $[-1, \bar{t}_j]$ to a fixed interval [–1, 1] *e. g.*:

$$\bar{u}(\bar{x},\bar{t}_j)=\rho_j\int_{-1}^{1}(1-\theta)^{-\mu}\left(\hat{\mathcal{L}}\{\bar{t}_j,s_j(\theta),\bar{u}[\bar{x},s_j(\theta)]\}+\hat{\mathcal{F}}\{(\bar{t}_j,s_j(\theta),\bar{u}[\bar{x},s_j(\theta)]\}\right)\mathrm{d}\theta+\bar{f}(\bar{x},\bar{t}_j)+\bar{\phi}(\bar{x},\bar{t}_j) \quad (9)$$

We first use $\bar{u}^j(x)$ to approximate the function $\bar{u}(\bar{x}, \bar{t}_j)$:

$$\bar{u}\left(\bar{x},\bar{t}\right)\approx\bar{u}^M\left(\bar{x},\bar{t}\right)=\sum_{j=0}^{M}\bar{u}^j\left(\bar{x}\right)F_j\left(\bar{t}\right) \quad (10)$$

where $F_j(\bar{t})$ are the Lagrange interpolation polynomials associated with collocation points $\{\bar{t}_j\}_{j=0}^{M}$.

Combining the Gauss quadrature formula and eqs. (10) and (9) can be approximated:

$$\bar{u}^j(\bar{x})=\bar{\rho}_j\sum_{k=0}^{L}(1-\theta_k)\left(\bar{\mathcal{L}}\bar{u}^M[\bar{x},s_j(\theta_k)]+\bar{\mathcal{F}}\left\{\bar{u}^M[\bar{x},s_j(\theta_k)]\right\}\right)\omega_k^{-\mu,0}+\bar{f}(\bar{x},\bar{t}_j)+\bar{\phi}(\bar{x},\bar{t}_j) \quad (11)$$

where $\{\theta_k\}_{k=0}^{L}$ is the set of the Jacobi-Gauss points corresponding to the weights $\{\omega_k^{-\mu,0}\}$on [–1, 1].

For the Jacobi collocation methods in space, eq. (11) holds at the Legendre-Gauss collocation points $\{\bar{x}_i\}_{i=0}^{N}$, use $\bar{u}_i^j$ to approximate the function $\bar{u}^j(\bar{x}_i)$:

$$\bar{u}^j(\bar{x})\approx\sum_{i=0}^{N}\bar{u}_i^jH_i(\bar{x}),\ \bar{u}(\bar{x},\bar{t})\approx\bar{u}_N^M(\bar{x},\bar{t})=\sum_{i=0}^{N}\sum_{j=0}^{M}\bar{u}_i^jH_i(\bar{x})F_j(\bar{t})$$

where $\{H_i(\bar{x})\}_{i=0}^{N}$ are the i th Lagrange interpolation polynomials associated with the Legendre-Gauss points $\{\bar{x}_i\}_{i=0}^{N}$. The discretized problem in space eq. (11), can be approximated:

$$\bar{u}_i^j=\rho_j\sum_{k=0}^{L}(1-\theta_k)\left(\bar{\mathcal{L}}\bar{u}_N^M\left[\bar{x}_i,s_j(\theta_k)\right]+\bar{\mathcal{F}}\left\{\bar{u}_N^M\left[\bar{x}_i,s_j(\theta_k)\right]\right\}\right)\omega_k^{-\mu,0}+\bar{f}(\bar{x}_i,\bar{t}_j)+\bar{\phi}(\bar{x}_i,\bar{t}_j) \quad (12)$$

Then the full-discreted solution of eq. (13) is to seek $\bar{u}_N^M(\bar{x},\bar{t})$ such that $\bar{u}_i^j$ satisfies the above collocation eq. (12) for $1\le i\le N-1$ and $0\le j\le M$.

Convergence analysis

In this section, we carry out the error estimations for the solution of the semi-discretized and full-discretized problems. Now we devote to provide a convergence analysis for the numerical scheme. The goal is to show that the rate of convergence is exponential.

Let $\bar{u}(\bar{x},\bar{t})$ be the solution of the continuous problem eq. (5), and let:

$$\bar{u}^M(\bar{x},\bar{t})=\sum_{m=0}^{M}\bar{u}^m(\bar{x})F_m(\bar{t})$$

be the time-discrete solution of eq. (11), for y and $m>1$. If $\bar{u}(\bar{x},\bar{t})$, $\bar{u}_{\bar{x}}(\bar{x},\bar{t})$, and:

$$\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})\in H^m_{\omega^{-\mu,-\mu}}(I)$$

then for M sufficiently large:

$$E_{L_\infty}\le\begin{cases}CM^{1/2-m}\left[U_1(\bar{x})+M^{-1/2}\log MF^*(\bar{x})\right]\left[U_1(\bar{x})+M^{-1/2}\log MF^*(\bar{x})\right]-1\le-\mu<-\dfrac{1}{2}\\ CM^{1-\mu-m}[\log MU_1(\bar{x})+M^{-1/2}F^*(\bar{x})],-\dfrac{1}{2}\le-\mu<\dfrac{1}{2}-\mu\end{cases} \quad (13a)$$

$$E_{L^2}\le\begin{cases}CM^{-m}\left[U_2(\bar{x})+M^{1/2-\kappa}U_1(\bar{x})+F^*(\bar{x})\right],-1\le-\mu<-\dfrac{1}{2}\\ CM^{-m}\log M\left[U_2(\bar{x})+M^{1-\mu-\kappa}U_1(\bar{x})+F^*(\bar{x})\right],-\dfrac{1}{2}\le-\mu<\dfrac{1}{2}-\mu\end{cases} \quad (13b)$$

for $\kappa \in (0,1-\mu)$, where C is a constant independent of M.

$$E_{L_\infty} = \| \bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t}) \|_{L^\infty},\ E_{L^2} = \| \bar{u}(\bar{x},\bar{t}) - \bar{u}^M(\bar{x},\bar{t}) \|_{L^2_{-\mu,-\mu}}$$

$$F^*(\bar{x}) = \max_{\bar{t},s\in[-1,1]} | \hat{\mathcal{F}}\{\bar{t},\bar{u}[\bar{x},s(\theta)]\} |_{H^{m,M}_{\omega^{-\mu,0}}(I)}$$

$$U_1(\bar{x}) = | \bar{u}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^c}} + | \bar{u}_{\bar{x}}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^c}} + | \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^c}}$$

$$U_2(x) = | \bar{u}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^{-\mu,-\mu}}} + | \bar{u}_{\bar{x}}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^{-\mu,-\mu}}} + | \bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) |_{H^{m,M}_{\omega^{-\mu,-\mu}}}$$

Using the boundary conditions:

$$\begin{gathered}\bar{u}(\bar{x}) = \int_{-1}^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j)\mathrm{d}\tau + \bar{\psi}_1(\bar{t}_j),\ \bar{u}^M_{\bar{x}}(\bar{x},\bar{t}_j) = \int_\xi^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j)\mathrm{d}\tau + \frac{\bar{\psi}_2(\bar{t}_j)-\bar{\psi}_1(\bar{t}_j)}{2} \\ \bar{u}_{\bar{x}}(\bar{x},\bar{t})\Big|_{\bar{x}=\xi} = \frac{\bar{\psi}_2(\bar{t})-\bar{\psi}_1(\bar{t})}{2},\ \bar{\psi}_i(\bar{t}) = \psi_i\left[\frac{T}{2}(1+\bar{t})\right],\ i=1,2\end{gathered} \tag{14}$$

Consequently, $\bar{u}^M(\bar{x},\bar{t})$ satisfies the following collocation equations for $0 \le j \le M$:

$$\bar{u}(\bar{x}) = \int_{-1}^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j)\mathrm{d}\tau + \bar{\psi}_1(\bar{t}_j),\ \bar{u}^M_{\bar{x}}(\bar{x},\bar{t}_j) = \int_\xi^{\bar{x}} \bar{u}^M_\tau(\tau,\bar{t}_j)\mathrm{d}\tau + \frac{\bar{\psi}_2(\bar{t}_j)-\bar{\psi}_1(\bar{t}_j)}{2} \tag{15}$$

Since $L \le M$ and:

$$\bar{u}^M(x,y) = \sum_{m=0}^{M} \bar{u}^m(x)F_m(y) \in P_L(y)$$

we have:

$$\int_{-1}^{1}(1-\theta)^{-\mu}\hat{\mathcal{L}}\{\bar{t}_j,s_j,\bar{u}^M[x,s_j(\theta)]\}\mathrm{d}\theta = \sum_{k=0}^{L}\hat{\mathcal{L}}\{\bar{t}_j,s_j,\bar{u}^M[x,s_j(\theta_k)]\}\omega_k^{-\mu,0}$$

Subtracting eq. (9) from eq. (11), and eq. (14) from eq. (13):

$$\begin{gathered}\bar{u}(\bar{x},\bar{t}_j) - \bar{u}^j(\bar{x}) = \frac{1}{\Gamma(\gamma)}\left(\frac{T}{2}\right)^\gamma \int_{-1}^{\bar{t}_j}(\bar{t}_j-s)^{-\mu}\left\{\hat{\mathcal{L}}[\bar{t}_j,s,\bar{u}(\bar{x},s)] - \hat{\mathcal{L}}[\bar{t},s,\bar{u}^M(\bar{x},s)]\right\}\mathrm{d}s + K_0(\bar{x},\bar{t}_j) \\ \bar{u}(\bar{x},\bar{t}_j) - \bar{u}^j(\bar{x}) = \int_{-1}^{\bar{x}}\bar{u}_\tau(\tau,\bar{t}_j)\mathrm{d}\tau - \int_{-1}^{\bar{x}}\bar{u}^M_\tau(\tau,\bar{t}_j)\mathrm{d}\tau \\ \bar{u}_{\bar{x}}(\bar{x},\bar{t}_j) - \bar{u}^M_{\bar{x}}(\bar{x},\bar{t}_j) = \int_\xi^{\bar{x}}\bar{u}_{\tau\tau}(\tau,\bar{t}_j)\mathrm{d}\tau - \int_\xi^{\bar{x}}\bar{u}^M_{\tau\tau}(\tau,\bar{t}_j)\mathrm{d}\tau\end{gathered}$$

$$K_0(\bar{x},\bar{t}_j) = \rho_j\sum_{k=0}^{L}\hat{\mathcal{F}}\{\bar{t}_j,s_j(\theta_k),\bar{u}[\bar{x},s_j(\theta_k)]\}\omega^{-\mu,0} - \rho_j\int_{-1}^{1}(1-\theta)^{-\mu}\hat{\mathcal{F}}\{\bar{t}_j,s_j(\theta),\bar{u}^M[\bar{x},s_j(\theta)]\}\mathrm{d}\theta \tag{16}$$

Using [9] (see p. 290, (5.4.38)) the definition of $|\cdot|_{H^{m,n}(I)}$ and Lipschitz condition:

$$\begin{gathered}|K_0(\bar{x},\bar{t}_j)| \le CM^{-m}|\hat{\mathcal{F}}\{\bar{t}_j,s_j,\bar{u}^M[\bar{x},s(\theta)]\}|_{H^{m,M}_{\omega^{-\mu,0}}(I)} \le \\ \le CM^{-m}\left(|\hat{\mathcal{F}}\{\bar{t}_j,s_j,\bar{u}[\bar{x},s(\theta)]\}|_{H^{m,M}_{\omega^{-\mu,0}}(I)} + \|\bar{u}^M - \bar{u}\|_{L^\infty(I)}|\right)\end{gathered} \tag{17}$$

Multiplying by $F(\bar{y})$ both sides of eq. (16) and summing from 0 to M for j:

$$I_M^{-\mu,-\mu}\overline{u}(\overline{x},\overline{t})-\overline{u}^M(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\rho(\overline{t})\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\left(\{\hat{\mathcal{L}}[\overline{t},s,\overline{u}(\overline{x},s)]-\hat{\mathcal{L}}[\overline{t},s,\overline{u}^M(\overline{x},s)]\}\right)+K_0(\overline{x},\overline{t})$$

$$I_M^{-\mu,-\mu}\overline{u}(\overline{x},\overline{t})-\overline{u}^M(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\left[\int_{-1}^{\overline{x}}\overline{u}_\tau(\tau,\overline{t})\mathrm{d}\tau-\int_{-1}^{\overline{x}}\overline{u}_\tau^M(\tau,\overline{t})\mathrm{d}\tau\right]$$

$$I_M^{-\mu,-\mu}\overline{u}_{\overline{x}}(\overline{x},\overline{t})-\overline{u}_{\overline{x}}^M(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\left[\int_{\xi}^{\overline{x}}\overline{u}_{\tau\tau}(\tau,\overline{t})\mathrm{d}\tau-\int_{\xi}^{\overline{x}}\overline{u}_{\tau\tau}^M(\tau,\overline{t})\mathrm{d}\tau\right] \tag{18}$$

where

$$K_0(\overline{x},\overline{t})=I_M^{-\mu,-\mu}K_0(\overline{x},\overline{t}_j)$$

Let:

$$E_0(\overline{x},\overline{t})=\overline{u}(\overline{x},\overline{t})-\overline{u}^M(\overline{x},\overline{t}),\ E_1(\overline{x},\overline{t})=\overline{u}_{\overline{x}}(\overline{x},\overline{t})-\overline{u}_{\overline{x}}^M(\overline{x},\overline{t}),\ E_2(\overline{x},\overline{t})=\overline{u}_{\overline{xx}}(\overline{x},\overline{t})-\overline{u}_{\overline{xx}}^M(\overline{x},\overline{t})$$

It follows from eq. (18) that:

$$E_0(\overline{x},\overline{t})=\rho(\overline{t})\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\hat{L}\left[\overline{t},s,E_0(\overline{x},s)\right]\mathrm{d}s+K_0(\overline{x},\overline{t})+K_1(\overline{x},\overline{t})+K_2(\overline{x},\overline{t})$$

$$E_0(\overline{x},\overline{t})=\int_{-1}^{\overline{x}}E_1(\tau,\overline{t})\mathrm{d}\tau+K_1(\overline{x},\overline{t})+K_3(\overline{x},\overline{t}),E_1(\overline{x},\overline{t})=\int_{\xi}^{\overline{x}}E_2(\tau,\overline{t})\mathrm{d}\tau+K_4(\overline{x},\overline{t})+K_5(\overline{x},\overline{t}) \tag{19}$$

where

$$K_4(\overline{x},\overline{t})=\overline{u}_{\overline{x}}(\overline{x},\overline{t})-I_M^{-\mu,-\mu}\overline{u}_{\overline{x}}(\overline{x},\overline{t}),\ K_5(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\int_{\xi}^{\overline{x}}E_2(\tau,\overline{t})\mathrm{d}\tau-\int_{\xi}^{\overline{x}}E_2(\tau,\overline{t})\tau.$$

Then:

$$\hat{\mathcal{L}}E_0(\overline{x},s)=(\overline{t}-s)\left[E_2(\overline{x},s)-\overline{\xi}E_0(\overline{x},s)\right],\ K_1(\overline{x},\overline{t})=\overline{u}(\overline{x},\overline{t})-I_M^{-\mu,-\mu}\overline{u}(\overline{x},\overline{t})$$

$$K_2(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\rho(\overline{t})\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\hat{\mathcal{L}}(\overline{t},s,E_0)\mathrm{d}s-\rho(\overline{t})\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\hat{\mathcal{L}}(\overline{t},s,E_0)\mathrm{d}s$$

$$K_3(\overline{x},\overline{t})=I_M^{-\mu,-\mu}\int_{-1}^{\overline{x}}E_1(\tau,\overline{t})\mathrm{d}\tau-\int_{-1}^{\overline{x}}E_1(\tau,\overline{t})\mathrm{d}\tau$$

Note that $\overline{L}E_0(\overline{x},\overline{t})=E_2(\overline{x},\overline{t})-\ \xi E_0(\overline{x},\overline{t})$, we have:

$$|E_0(\overline{x},\overline{t})|\le C\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\left[|E_0(\overline{x},s)|+|E_2(\overline{x},s)|\right]\mathrm{d}s+|K_0(\overline{x},\overline{t})|+|K_1(\overline{x},\overline{t})|+|K_2(\overline{x},\overline{t})|$$

$$|E_0(\overline{x},\overline{t})|\le\int_{-1}^{\overline{x}}|E_1(\tau,\overline{t})|\mathrm{d}\tau+|K_1(\overline{x},\overline{t})|+|K_3(\overline{x},\overline{t})|, \tag{20}$$

$$|E_1(\overline{x},\overline{t})|\le\int_{\xi}^{\overline{x}}|E_2(\tau,\overline{t})|\,\mathrm{d}\tau+K_4(\overline{x},\overline{t})|+|K_5(\overline{x},\overline{t})|$$

which gives:

$$|E_0(\overline{x},\overline{t})|\le C\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\,|E_0(\overline{x},s)|\mathrm{d}s+\sum_{i=0,1,4,6,7}|K_i(\overline{x},\overline{t})|$$

$$K_6(\overline{x},\overline{t})=\overline{u}_{\overline{xx}}(\overline{x},\overline{t})-I_M^{-\mu,-\mu}\overline{u}_{\overline{xx}}(\overline{x},\overline{t}) \tag{21}$$

$$K_7(\overline{x},\overline{t})=CI_M^{-\mu,-\mu}\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\,|E_0(\overline{x},s)|\mathrm{d}s-C\int_{-1}^{\overline{t}}(\overline{t}-s)^{-\mu}\,|E_0(\overline{x},s)\mathrm{d}s$$

Using the Gronwall inequality in [10] (see *Lemma 7.1.1*), we deduce:

$$\| E_0(\bar{x},\bar{t})\|_{L^\infty(I)} \le \sum_{i=0,1,4,6,7} \left\|K_i(\bar{x},\bar{t})\right\|_{L^\infty(I)}, \|E_0(\bar{x},\bar{t})\|_{L^2_{-\mu,-\mu}(I)} \le \sum_{i=0,1,4,6,7} \left\|K_i(\bar{x},\bar{t})\right\|_{L^2_{-\mu,-\mu}}$$

Using L^∞ and weighted L^2 error bounds, see [10] and eq. (9), from eq. (17):

$$\| K_0(\bar{x},\bar{t})\|_{L^\infty(I)} \le \begin{cases} CM^{\frac{1}{2}-\mu-m}\left(\max\limits_{\bar{t},s\in[-1,1]} | \hat{\mathcal{F}}\{\bar{t},s,\bar{u}[\bar{x},s(\theta)]\}\right)|_{H^{m,M}_{\omega^{-\mu,0}}(I)} + \sum\limits_{i=1,4,6,7}\left\|K_i(\bar{x},\bar{t})\right\|_{L^\infty(I)} \\ -\dfrac{1}{2} < -\mu < 0, \\ CM^{-m}\log M\left(\max\limits_{\bar{t},s\in[-1,1]} \hat{\mathcal{F}}\{\bar{t},\bar{u}[\bar{x},s(\theta)]\}\right)|_{H^{m,M}_{\omega^{-\mu,0}}(I)} + \sum\limits_{i=1,4,6,7}\left\|K_i(\bar{x},\bar{t})\right\|_{L^\infty(I)} \\ -1 \le -\mu < \dfrac{1}{2} \end{cases} \tag{22}$$

$$\| K_1(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{1/2-m} \,|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}} \\ CM^{1-\mu-m}\log M\,|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}} \end{cases} \tag{23}$$

$$\| K_4(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{1/2-m} \,|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}},\ -1\le -\mu < -\dfrac{1}{2} \\ CM^{1-\mu-m}\log M\,|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}},\ \text{otherwise} \end{cases} \tag{24}$$

$$\| K_6(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{1/2-m} \,|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}},\ -1\le -\mu < -\dfrac{1}{2} \\ CM^{1-\mu-m}\log M\,|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^c}},\ \text{otherwise} \end{cases} \tag{25}$$

By [11], we get:

$$\|K_0(\bar{x},\bar{t})\|_{L^2_{-\mu,-\mu}} \le CM^{-m}\,|\bar{u}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{-\mu,-\mu}}}$$

$$\left\|K_4(\bar{x},\bar{t})\right\|_{L^2_{-\mu,-\mu}} \le CM^{-m}\,|\bar{u}_{\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{-\mu,-\mu}}},\ \left\|K_6(\bar{x},\bar{t})\right\|_{L^2_{-\mu,-\mu}} \le CM^{-m}\,|\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t})|_{H^{m,M}_{\omega^{-\mu,-\mu}}} \tag{26}$$

Finally, it follows from [10] (see p.153 and *Lemma 3.5*,), and [12], see eq. (9):

$$\| K_7(\bar{x},\bar{t})\|_{L^\infty} \le \begin{cases} CM^{-\kappa}\log M\,\||E_0(\bar{x},s)\||_{L^\infty},\ -1\le -\mu < -\dfrac{1}{2}, \kappa\in(0,1-\mu) \\ CM^{\frac{1}{2}-\mu-\kappa}\,\||E(x,s)\||_{L^\infty},\ -\dfrac{1}{2} \le -\mu < \dfrac{1}{2}-\mu, \kappa\in(0,1-\mu) \end{cases} \tag{27}$$

$$\| K_7(\bar{x},\bar{t})\|_{L^2_{-\mu,-\mu}} = \|(I_M^{-\mu,-\mu} - I)(\mathcal{M}\,|E_0(\bar{x},s)| - \mathcal{T}_M\,|E_0(\bar{x},s)\||_{L^2_{-\mu,-\mu}} \le CM^{-\kappa}\,\||E_0(\bar{x},s)\||_{L^\infty} \tag{28}$$

Equations (23)-(25) and eq. (27) leads to eq. (13a). In a similar way, eqs. (26)-(28) leads to eq. (13b).

Let $\bar{u}^M(\bar{x},\bar{t})$ be the time-discrete solution of eq. (11) and let $\bar{u}^M_N(\bar{x},\bar{t})$ be the solution of the full-discrete problem of eq. (12) with boundary conditions eq. (6). If $\bar{u}^M(\bar{x},\bar{t})$, $\bar{u}^M_{\bar{x}}(\bar{x},\bar{t})$, and:

$$\bar{u}^M_{\bar{x}\bar{x}}(\bar{x},\bar{t}) \in H^{n,N}_{\omega^{0,0}}(I)$$

for x and $n > 1$, then the following error estimation holds:

$$\|\bar{u}^M(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})\|_{L^\infty}\le\begin{cases}CN^{3/4-n}\log MV_1, & -1\le-\mu<-\dfrac{1}{2}\\ CN^{3/4-n}M^{\frac{1}{2}-\mu}V_1, & \text{otherwise}\end{cases}$$

$$\left\|\bar{u}^M(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})\right\|_{L^2}\le CN^{-n}V_1, V_1=\max_{\bar{t}\in[-1,1]}\left[\left|\bar{u}^M(\bar{x},\bar{t})\right|_{H^{n,N}}+\left|\bar{u}_{\bar{x}}^M(\bar{x},\bar{t})\right|_{H^{n,N}}+\left|\bar{u}_{\bar{x}\bar{x}}^M(\bar{x},\bar{t})\right|_{H^{n,N}}\right] \quad (29)$$

where C is independent of N and M. By subtracting eq. (11) from eq. (12):

$$\bar{u}^j(\bar{x}_i)-\bar{u}_i^j\le\sum_{m=0}^{M}C_m^j\left|\bar{\mathcal{L}}\bar{u}^m(\bar{x}_i)-\bar{\mathcal{L}}\sum_{n=0}^{N}\bar{u}_n^m H_n(\bar{x}_i)\right|$$

$$C_m^j=C\frac{1}{\Gamma(\gamma)}\left(\frac{T}{2}\right)^{\gamma}\int_{-1}^{\bar{t}_j}(\bar{t}_j-s)^{-\mu}F_m(\bar{t})\mathrm{d}s \quad (30)$$

Multiplying by $H_i(\bar{x})$ both sides of eq. (29) and summing from 0 to N for i, and noting that:

$$\bar{\mathcal{L}}\sum_{n=0}^{N}\bar{u}_n^m H_n(\bar{x})\in P_N(\bar{x}),\ \text{then}\ I_N\bar{\mathcal{L}}\sum_{n=0}^{N}\bar{u}_n^m H_n(\bar{x}_i)=\bar{\mathcal{L}}\sum_{n=0}^{N}\bar{u}_n^m H_n(\bar{x}),\ \text{it yield that:}$$

$$I_N\bar{u}^j(\bar{x})-\sum_{i=0}^{N}\bar{u}_i^j H_i(\bar{x})\le\sum_{m=0}^{M}C_m^j\left|I_N\bar{\mathcal{L}}\bar{u}^j(\bar{x})-\bar{\mathcal{L}}\sum_{n=0}^{N}\bar{u}_n^j H_n(\bar{x})\right| \quad (31)$$

Let:

$$e^{j\prime\prime}(\bar{x})=\bar{u}^{j\prime\prime}(\bar{x})-\sum_{i=0}^{N}\bar{u}_i^j H_i''(\bar{x}),\ J_2(\bar{x})=\bar{u}^{j\prime}(\bar{x})-I_N\bar{u}^{j\prime}(\bar{x}),\ J_3(\bar{x})=\bar{u}^{j\prime\prime}(\bar{x})-I_N\bar{u}^{j\prime\prime}(\bar{x})$$

$$e(\bar{x},\bar{t}_j)=e^j(\bar{x})=\bar{u}^j(\bar{x})-\sum_{i=0}^{N}\bar{u}_i^j H_i(\bar{x}),\ e^{j\prime}(\bar{x})=\bar{u}^{j\prime}(\bar{x})-\sum_{i=0}^{N}\bar{u}_i^j H_i'(\bar{x}),\ J_1(\bar{x})=\bar{u}^j(\bar{x})-I_N\bar{u}^j(\bar{x})$$

Thus we have:

$$|e^j|\le C(|J_1(\bar{x})|+|J_2(\bar{x})|+|J_3(\bar{x})|) \quad (32)$$

Using L^∞ and L^2 error bounds for the interpolation polynomials in [9]:

$$\|J_1\|_{L^\infty}\le CN^{3/4-n}\left|\bar{u}^j\right|_{H^{n,N}},\|J_2\|_{L^\infty}\le CN^{3/4-n}\left|\bar{u}^{j\prime}\right|_{H^{n,N}},\|J_3\|_{L^\infty}\le CN^{3/4-n}\left|\bar{u}^{j\prime\prime}\right|_{H^{n,N}}$$

$$\|J_1\|_{L^2}\le CN^{-n}\left|\bar{u}^j\right|_{H^{n,N}},\|J_2\|_{L^2}\le CN^{-n}\left|\bar{u}^{j\prime}\right|_{H^{n,N}},\|J_3\|_{L^2}\le CN^{-n}\left|\bar{u}^{j\prime\prime}\right|_{H^{n,N}}$$

$$\bar{u}^M(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})=\sum_{j=0}^{M}\left[u^j(\bar{x})-\sum_{i=0}^{N}u_i^j H_i(\bar{x})\right]F_j(\bar{t})=\sum_{j=0}^{M}e(\bar{x},\bar{t}_j)F_j(\bar{t})=I_M^{-\mu,-\mu}e(\bar{x},\bar{t}) \quad (33)$$

Here, we have [11, 12]:

$$\|u^M(\bar{x},\bar{t})-u_N^M(\bar{x},\bar{t})\|_{L^2_{-\mu,-\mu}}\le C\max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})| \quad (34)$$

$$\|\bar{u}^M(\bar{x},\bar{t})-\bar{u}_N^M(\bar{x},\bar{t})\|_{L^\infty}\le\begin{cases}C\log M\max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})|, & -1\le-\mu<-\dfrac{1}{2}\\ CM^{\frac{1}{2}-\mu}\max_{\bar{t}\in[-1,1]}|e(\bar{x},\bar{t})|, & \text{otherwise}\end{cases} \quad (35)$$

By using eqs. (32)-(35) and previous inequality, we obtain eq. (29). Thus, the proof is completed.

Let $\bar{u}(\bar{x},\bar{t})$ be the solution of the continuous problem (5) and let $\bar{u}_N^M(\bar{x},\bar{t})$ be the solution of the full-discrete problem eq. (12) with the initial condition (1) and boundary conditions (2). If $\bar{u}(\bar{x},\bar{t})$, $\bar{u}_{\bar{x}}(\bar{x},\bar{t})$ and $\bar{u}_{\bar{x}\bar{x}}(\bar{x},\bar{t}) \in H^{n,N}_{\omega\alpha_x,\beta_x} \otimes H^{m,N}_{\omega\alpha_y,\beta_y}$, for x and y with n, $m > 1$, then for M and N sufficiently large, for $\kappa \in (0, 1 - \mu)$, we have the following error estimations:

$$\| \bar{u}(\bar{x},\bar{t}) - \bar{u}_N^M(\bar{x},\bar{t}) \|_{L^\infty} \le \begin{cases} CM^{1/2-m}\left(U_1(\bar{x}) + M^{-1/2}\log MF^*(\bar{x})\right) + CN^{3/4-n}\log MV_1 \\ \quad -1 \le -\mu < -\dfrac{1}{2} \\ CM^{1-\mu-m}\left(\log MU_1(\bar{x}) + M^{-1/2}F^*(\bar{x})\right) + CN^{3/4-n}M^{1/2-\mu}V_1 \\ \quad -\dfrac{1}{2} \le -\mu < \dfrac{1}{2} - \mu \end{cases}$$

$$\left\| \bar{u}(\bar{x},\bar{t}) - \bar{u}_N^M(\bar{x},\bar{t}) \right\|_{L^2_{-\mu,-\mu}} \le \begin{cases} CM^{-m}\left[U_2 + M^{1/2-\kappa}U_1 + F^*(\bar{x})\right] + CN^{-n}V_1 \\ \quad -1 \le -\mu < -\dfrac{1}{2} \\ CM^{-m}\log M\left[U_2 + M^{1-\mu-\kappa}U_1 + F^*(\bar{x})\right] + CN^{-n}V_1 \\ \quad -\dfrac{1}{2} \le -\mu < \dfrac{1}{2} - \mu \end{cases} \tag{36}$$

Numerical experiments

We consider the following fractional Klein-Gordon equation:

$$D_t^\gamma u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u^3(x,t) + f(x,t), \quad 1 < \gamma < 2$$

where

$$f(x,t) = \left[\frac{2t^{2-\gamma}}{\Gamma(2-\gamma)} + t^2\pi^2\right]\cos(\pi x) + t^6\cos^3(\pi x)$$

with initial condition $u(x, 0) = 0$ and $u_t(x, 0) = 0$. The exact solution is $u(x, 0) = t^2\cos(\pi x)$.

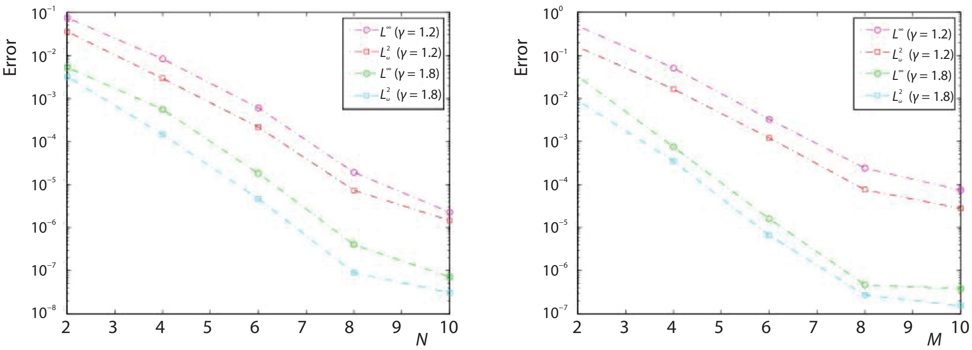

Figure 1. The variation of the error with the increase of the collection points

As shown in fig. 1 shows, that the error has the exponential convergence due to the variation of the error with (N, M) in the sense of the spaces $\|\cdot\|_{L_\infty}$ and $\|\cdot\|_{L^2_\omega}$.

Conclusion

In this work, the time-space spectral collocation method for time fractional Klein-Gordon equation was proposed. The equation holds at the Jacobi spectral collocation points in the time and space direction and the full discrete form is obtained in detail. This method is easy to deal with the non-linear situations. The numerical solution with higher accuracy can be obtained with fewer points.

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