A PERIODIC SOLUTION FOR THE LOCAL FRACTIONAL BOUSSINESQ EQUATION ON CANTOR SETS

by

Xiu-Rong GUO[a,b]*, Gui-Lei CHEN[a], Mei GUO[c], Zheng-Tao LIU[c]

[a] Basic Courses, Shandong University of Science and Technology, Tai'an, China
[b] College of Mathematics, China University of Mining and Technology, XuZhou, China
[c] Shandong Tonghui Architectural Design Co. Ltd., Taian, China

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In this paper, the periodic solution for the local fractional Boussinesq equation can be obtained in the sense of the local fractional derivative. It is given by applying direct integration with symmetry condition. Meanwhile, the periodic solution of the non-differentiable type with generalized functions defined on Cantor sets is analyzed. As a result, we have a new point to look the local fractional Boussinesq equation through the local fractional derivative theory.

Key words: *local fractional derivative, local fractional Boussinesq equation, periodic solution*

Introduction

It is known that fractional derivative can be better describing things than integral order derivative from natural science, such as particle physics, electrical systems, mathematical physics [1-3]). Many researchers proposed some effective methods to deal with them, such as the homotopy analysis method [4], the Lie group method [5-7], and others. However, the local fractional derivative only have a litter results to be known. For example, the local fractional Laplace series expansion method [8], and the local fractional Riccati differential equation method [9], and so on.

In this paper, we consider the local fractional Boussinesq equation of fractal long water waves of (1+1)-fractal dimensional space [10], *e. g*:

$$\frac{\partial^{2\varepsilon}\Omega(\chi,\tau)}{\partial\tau^{2\varepsilon}}-\alpha\frac{\partial^{4\varepsilon}\Omega(\chi,\tau)}{\partial\chi^{2\varepsilon}\partial\tau^{2\varepsilon}}-\beta\frac{\partial^{2\varepsilon}\Omega(\chi,\tau)}{\partial\chi^{2\varepsilon}}=0 \tag{1}$$

where $\Omega(\chi,\tau)$ is the fractal wave function on space variable, χ, and time variable, τ, while α and β are two constants, and ε is the fractal dimension.

To the best of our knowledge, it has not been reported in other places. The aim of this paper is to obtain the periodic solution of local fractional Boussinesq equation with symmetry condition.

* Corresponding author, e-mail: springming007@126.com

Preliminaries of local fractional calculus

In this section, some definitions and prosperities of the local fractional derivative and integral for the non-differential functions can be given by [11-15].

Definition 1. Suppose that $\Theta(\zeta) \in C_\varepsilon$ and $0 < \varepsilon \le 1$. For $\delta > 0$ and $0 < |\zeta - \zeta_0| < \delta$, the limit [10]:

$$D^{(\varepsilon)}\Theta(\zeta_0) = \frac{\mathrm{d}^{(\varepsilon)}\Theta(\zeta)}{\mathrm{d}\zeta^\varepsilon}\Big|_{\zeta=\zeta_0} = \lim_{\zeta\to\zeta_0} \frac{\Delta^\varepsilon\left[\Theta(\zeta) - \Theta(\zeta_0)\right]}{(\zeta - \zeta_0)^\varepsilon} \tag{2}$$

exists and is finite, where $\Delta^\varepsilon[\Theta(\zeta) - \Theta(\zeta_0)] \cong \Gamma(1+\varepsilon)[\Theta(\zeta) - \Theta(\zeta_0)]$ and C_ε is a set of the non-differentiable functions.

Some special functions defined on Cantor sets are listed as follows [10]:

$$\sin_\varepsilon(\mu^\varepsilon) = \frac{E_\varepsilon(i^\varepsilon\mu^\varepsilon) - E_\varepsilon(-i^\varepsilon\mu^\varepsilon)}{2i^\varepsilon} = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{(2k+1)\varepsilon}}{\Gamma[1+(2k+1)\varepsilon]} \tag{3}$$

$$\cos_\varepsilon(\mu^\varepsilon) = \frac{E_\varepsilon(i^\varepsilon\mu^\varepsilon) + E_\varepsilon(-i^\varepsilon\mu^\varepsilon)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{2k\varepsilon}}{\Gamma(1+2k\varepsilon)} \tag{4}$$

where i^ε is an imaginary unit of a fractal set.

Suppose that $\varphi(\zeta) \in C_\varepsilon(a,b)$. Then, we define the local fractional integral of $\varphi(\zeta)$ of order $\varepsilon(0 < \varepsilon \le 1)$ by [10]:

$${}_aI_b^{(\varepsilon)}\varphi(\zeta) = \frac{1}{\Gamma(1+\varepsilon)}\int_a^b \varphi(\zeta)(\mathrm{d}\zeta)^\varepsilon = \frac{1}{\Gamma(1+\varepsilon)} \lim_{\Delta\zeta_k\to 0} \sum_{k=0}^{N-1} \varphi(\zeta_k)(\Delta\zeta_k)^\varepsilon \tag{5}$$

where $\Delta\zeta_k = \zeta_{k+1} - \zeta_k, (k = 0,1,\cdots,N)$ with $a = \zeta_0 < \zeta_1 < \cdots < \zeta_{N-1} < \zeta_N = b$.

For the details of the local fractional derivatives integrals, see[10].

Periodic solutions of the non-differentiability

In this section, we will search for periodic wave solutions of the non-differentiable of eq. (1) with symmetry condition.

First, using the following fractal wave transformation [9]:

$$\Phi(\vartheta) = \Omega(\chi,\tau), \vartheta^\varepsilon = \chi^\varepsilon + \varpi^\varepsilon\tau^\varepsilon \tag{6}$$

where $\lim_{\varepsilon\to 1}\vartheta = \chi + \varpi\tau$ with the wave speed ϖ.

As a result, eq. (1) leads to:

$$\varpi^{2\varepsilon}\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} - \beta\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} + \alpha\varpi^{2\varepsilon}\frac{\partial^{4\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{4\varepsilon}} = 0 \tag{7}$$

Meanwhile, there is a symmetry condition:

$$\Phi(\vartheta) = \Phi(-\vartheta), \vartheta \in (-\infty,+\infty) \tag{8}$$

Let $\vartheta \ge 0$. Finding the local fractional integral of eq. (7) with respect to ϑ, we have:

$$\alpha\varpi^{2\varepsilon}\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} + \varpi^{2\varepsilon}\Phi(\vartheta) - \beta\Phi(\vartheta) = C_1, \quad \vartheta \ge 0 \tag{9}$$

where C_1 is a constant. When $C_1 = 0$, eq. (9) becomes:

$$\alpha\varpi^{2\varepsilon}\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} + \varpi^{2\varepsilon}\Phi(\vartheta) - \beta\Phi(\vartheta) = 0, \quad \vartheta \geq 0$$

Thus, we have

$$\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} + \left(\frac{1}{\alpha} - \frac{\beta}{\alpha\varpi^{2\varepsilon}}\right)\Phi(\vartheta) = 0, \quad \vartheta \geq 0$$

which can be rewritten in the form:

$$\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} + k^2\Phi(\vartheta) = 0, \quad \vartheta \geq 0 \tag{10}$$

where $k^2 = (1/\alpha - \beta/\alpha\varpi^{2\varepsilon})$ and it need to satisfy the constraint condition:

$$\frac{1}{\alpha} - \frac{\beta}{\alpha\varpi^{2\varepsilon}} \geq 0 \tag{11}$$

In what following, we construct the periodic solutions of the non-differentiable type under the constraint condition (11).

Making use of:

$$\Phi(\vartheta) = \cos_\varepsilon(k\vartheta^\varepsilon), \quad \frac{\partial^\varepsilon\Phi(\vartheta)}{\partial\vartheta^\varepsilon} = -k\sin_\varepsilon(k\vartheta^\varepsilon), \quad \frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} = -k^2\cos_\varepsilon(k\vartheta^\varepsilon) \tag{12}$$

we find a periodic solution of the non-defferentiable type of eq. (1), that is:

$$\Phi(\vartheta) = \cos_\varepsilon(k\vartheta^\varepsilon), \quad \vartheta \geq 0 \tag{13}$$

Using the symmetry condition (8), yields:

$$\Phi(\vartheta) = \cos_\varepsilon\left(k\left|\vartheta^\varepsilon\right|\right), \quad \vartheta \in (-\infty, +\infty) \tag{14}$$

in the whole interval.

Hence, the periodic solution of the non-differentiable type of the LBE in the form:

$$\Omega(\chi,\tau) = \cos_\varepsilon\left[\left(\frac{1}{\alpha} - \frac{\beta}{\alpha\varpi^{2\varepsilon}}\right)\left|\chi^\varepsilon + \varpi^\varepsilon\tau^\varepsilon\right|\right] \tag{15}$$

In a similar way, the periodic function solution of the non-differentiable type under the constraint condition (11) was also constructed:

First of all, we suppose:

$$\Phi(\vartheta) = \sin_\varepsilon(k\vartheta^\varepsilon), \quad \frac{\partial^\varepsilon\Phi(\vartheta)}{\partial\vartheta^\varepsilon} = -k\cos_\varepsilon(k\vartheta^\varepsilon)$$

and

$$\frac{\partial^{2\varepsilon}\Phi(\vartheta)}{\partial\vartheta^{2\varepsilon}} = -k^2\cos_\varepsilon(k\vartheta^\varepsilon) - k^2\sin_\varepsilon(k\vartheta^\varepsilon) \tag{16}$$

Therefore, we find a periodic solution of the non-defferentiable type of eq. (1), that is:

$$\Phi(\vartheta) = \sin_\varepsilon(k\vartheta^\varepsilon), \quad \vartheta \geq 0 \tag{17}$$

Using the symmetry condition (11), it follows that:

$$\Phi(\vartheta)=\sin_{\varepsilon}\left(k\left|\vartheta^{\varepsilon}\right|\right), \quad \vartheta\in(-\infty,+\infty) \tag{18}$$

Therefore, the periodic solution of the non-differentiable type of the LBE in the form:

$$\Omega(\chi,\tau)=\sin_{\varepsilon}\left[\left(\frac{1}{\alpha}-\frac{\beta}{\alpha\varpi^{2\varepsilon}}\right)\left|\chi^{\varepsilon}+\varpi^{\varepsilon}\tau^{\varepsilon}\right|\right] \tag{19}$$

As a result, the general periodic solution of combination of trigonometric functions of the non-differentiable type of the LBE in the form:

$$\Omega(\chi,\tau)=\cos_{\varepsilon}\left[\left(\frac{1}{\alpha}-\frac{\beta}{\alpha\varpi^{2\varepsilon}}\right)\left|\chi^{\varepsilon}+\varpi^{\varepsilon}\tau^{\varepsilon}\right|\right]+\sin_{\varepsilon}\left[\left(\frac{1}{\alpha}-\frac{\beta}{\alpha\varpi^{2\varepsilon}}\right)\left|\chi^{\varepsilon}+\varpi^{\varepsilon}\tau^{\varepsilon}\right|\right] \tag{20}$$

Conclusion

In our task, with the help of the local fractional derivatives and symmetry condition, a periodic wave solution of the local fractional Boussinesq equation was obtained on Cantor sets. It can help us better understand the physical phenomena and give a new perspective to look to wave theory. For other types solutions, such as rational solutions, solitary wave solutions and traveling wave solutions of the non-differentiable type, we will further discuss them in the forthcoming day's future work.

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Nomenclature

$C_{\varepsilon}(a,b)$ – Cantor set, [–]

Greek symbols

α – free constant, [–]
β – free constant, [–]
ε – fractal dimension, [–]
τ – time variable, [s]
χ – spatial variable, [m]
ϖ – wave speed, [ms^{-1}]

References

[1] Tarasov, V., *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media,* Higher Education Press Springer, New York, USA, 2011
[2] Liu, J., *et al.,* Non-Linear Dynamics and Exact Solutions for the Variable-Coefficient Modified Korteweg deVries Equation, *Zeitschrift für Naturforschung A*, *73* (2018), 2, pp. 143-149
[3] Appleby, M., *et al,.* Constructing Exact Symmetric Informationally Complete Measurements from Numerical Solution, *Journal of Physics A: Mathematical and Theoretical, 51* (2018), 16, pp. 165302
[4] Abbasbandy, S., The Application of Homotopy Analysis Method to Nonlinear Equations Arising in Heat Transfer, *Physics Letters A, 360* (2006), 1, pp. 109-113
[5] Zhang, Y., *et al.,* Lie Symmetry Analysis, Lie-Baklund Symmetries, Explicit Solutions, and Conservation Laws of Drinfeld-Sokolov-Wilson System, *Boundary Value Problems*, *2017* (2017), 1, pp. 154
[6] Bekir, A., *et al.,* Exact Solutions of Nonlinear Fractional Differential Equations by (G'/G)-Expansion Method, *Chinese Physics B*, *22* (2013), 11, pp. 140-145
[7] Liu, J. *et al.,* Analytical Study of Exact Solutions of the Nonlinear Korteweg deVries Equation with Space-time Fractional Derivatives, *Modern Physics Letters B*, *32* (2018), 2, 1850012

[8] Hemeda, A. A., *et al.,* Local Fractional Analytical Methods for Solving Wave Equations with Local Fractional Derivative, *Mathematical Methods in the Applied Sciences*, *41* (2018), 6, pp. 2515-2529
[9] Yang, X. J., *et al.,* Exact Travelling Wave Solutions for the Local Fractional Two-Dimensional Burgers-Type Equations, *Computers & Mathematics with Applications*, *73* (2017), 2, pp. 203-210
[10] Yang, X. J*, et al., Local Fractional Integral Transforms and Their Applications,* Academic Press, New York, USA, 2015
[11] Kumar, D., *et al.,* An Effective Computational Approach for Solving Local Fractional Telegraph Equations, *Nonlinear Science Letert A*, *8* (2017), 2, pp. 200-206
[12] Singh, J., *et al.,* A Reliable Algorithm for Local Fractional Tricomi Equation Arising in Fractal Transonic Flow, *Entropy*, *18* (2016), 6, pp. 206-214
[13] Yang, X., *et al,* On Exact Traveling-Wave Solutions for Local Fractional Korteweg-De Vries Equation, *Chaos, 26* (2016), 8, pp. 084312
[14] Yang, X. J., *et al.*, New Rheological Models within Local Fractional Derivative, *Romanian Reports in Physics*, *69* (2017), 3, pp.1-8
[15] Yang, X. J., *et al.*, A New Family of the Local Fractional PDEs, *Fundamenta Informaticae*, *151* (2017), 1-4, pp. 63-75

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