A LOCAL FRACTIONAL HOMOTOPY PERTURBATION METHOD FOR SOLVING THE LOCAL FRACTIONAL KORTEWEG-DE VRIES EQUATIONS WITH NON-HOMOGENEOUS TERM

by

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In this paper, a local fractional homotopy perturbation method is presented to solve the boundary and initial value problems of the local fractional Korteweg-de Vries equations with non-homogeneous term. In order to demonstrate the validity and reliability of the method, two types of the Korteweg-de Vries equations with non-homogeneous term are proposed.

Key words: *local fractional homotopy perturbation method, local fractional derivate, local fractional Korteweg-de Vries equation*

Introduction

The Korteweg-de Vries (KdV) equations and its relatives are widely applied for the description of non-linear waves in many branches of physics and engineering, such as electrodynamics, elastic media, traffic flow, fluid dynamics [1-7]. These equations are often too complicated to be solved exactly and even if an exact solution is obtained. The required calculations may be too complicated. A lot of research methods have been applied to derive the exact solutions of these equations, such as the homotopy analysis method [8], the variational iteration method [9], the functional variable method [10].

The local fractional derivative [11, 12] is the best method for describing the non-differential problems defined on Cantor sets. In those papers of Yang *et al*. [13] they have applied the local fractional differential equations on the Cantor fractal sets to describe many natural phenomena in fractal-like media, such as the local fractional KdV equation, the local fractional Tricomi equation [14], the local fractional heat conduction equation [15] and so on. Many methods have been developed to solve these local fractional differential equations, such as the local fractional variational iteration method [16], the Yang-Laplace transform method [17], the local fractional Fourier series method [18], the variational iteration transform method [19]and others [20-23].

Mathematical fundamentals

In this section, we introduce some mathematical preliminaries of the local fractional calculus theory in fractal space for our subsequent discussions [11, 12].

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Definition 1. Suppose that there is [12]:

$$\left|u(x)-u(x_0)\right|<\varepsilon^\alpha \tag{1}$$

with $|x-x_0|<\delta$, for $\varepsilon, \delta>0$ and $\varepsilon, \delta \in R$, then $u(x)$ is called local fractional continuous at $x=x_0$ and it is denoted by $\lim_{x\to x_0} u(x)=u(x_0)$.

Definition 2. Suppose that the function $u(x)$ is satisfied the eq. (1) for $x \in (a, b)$ it is called local fractional continuous on the interval (a, b):

$$u(x)\in C_\alpha(a,b) \tag{2}$$

Definition 3. In fractal space, let $u(x) \in C_\alpha(a, b)$, the local fractional derivative of $u(x)$ of order α at $x = x_0$ is given [12]:

$$\mathrm{D}_x^{(\alpha)}u(x_0)=u^{(\alpha)}\left(x_0\right)=\frac{\mathrm{d}^\alpha u(x)}{\mathrm{d}x^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha[u(x)-u(x_0)]}{(x-x_0)^\alpha} \tag{3}$$

where

$$\Delta^\alpha[u(x)-u(x_0)]\cong\Gamma(1+\alpha)\Delta[u(x)-u(x_0)]$$

The local fractional derivative of high order and the local fractional partial derivative of high order are defined, respectively, in the following forms [11, 12]:

$$u^{(k\alpha)}(x)=\overbrace{\mathrm{D}_x^{(\alpha)}\ldots\mathrm{D}_x^{(\alpha)}}^{k\ \text{times}}u(x) \tag{4}$$

$$u_x^{(k\alpha)}(x,y)=\frac{\partial^{k\alpha}}{\partial x^{k\alpha}}u(x,y)=\overbrace{\frac{\partial^\alpha}{\partial x^\alpha}\ldots\frac{\partial^\alpha}{\partial x^\alpha}}^{k\ \text{times}}u\left(x,y\right) \tag{5}$$

Definition 4. [11, 12] In fractal space, let $u(x) \in C_\alpha(a, b)$, the local fractional integral of $u(x)$ of order α in the interval $[a, b]$ is defined:

$${}_aI_b^{(\alpha)}u(x)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_a^b u(t)(\mathrm{d}t)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}u(t_j)(\Delta t_j)^\alpha \tag{6}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j,\ldots\}$ and $[t_j, t_{j+1}]$, $j = 0,\ldots N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

The local fractional homotopy perturbation method

In this section, we shall present the process of the local fractional homotopy perturbation method [24] to derive exact solutions of the local fractional KdV equations with non-homogeneous term.

Firstly, we consider the following local fractional KdV equations with non-homogeneous term on fractal set, which is given in the following form:

$$\alpha u_t^{(\alpha)}+\beta uu_x^{(\alpha)}+u_x^{(3\alpha)}=f(x,t) \tag{7}$$

where $f(x, t)$ is a non-homogeneous term.

By using the homotopy perturbation method and according to eq. (7), we construct the following homotopy:

$$(1-p^\alpha)\left[v_x^{(3\alpha)}-u_0\right]+p^\alpha\left[v_x^{(3\alpha)}+\alpha v_t^{(\alpha)}+\beta vv_x^{(\alpha)}-f(x,t)\right]=0 \tag{8}$$

where $v = v(x, t, p)$.

This is:

$$v_x^{(3\alpha)} = u_0 - p^\alpha \left[u_0 + \alpha v_t^{(\alpha)} + \beta v v_x^{(\alpha)} - f(x,t) \right] \tag{9}$$

where $p \in [0, 1]$ and where $u_0(x, t)$ is a preliminary approximation of $u_0(x, t)$.

Applying the local fractional triple integral ${}_0I_x^{(3\alpha)}(\bullet)$ on both sides of eq. (9), we obtain:

$$v = \tilde{v} + {}_0I_x^{(3\alpha)} u_0 - p^\alpha \, {}_0I_x^{(3\alpha)} \left[u_0 + \alpha v_t^{(\alpha)} + \beta v v_x^{(\alpha)} - f(x,t) \right] \tag{10}$$

where $\tilde{v}(x, t)$ is derived from the initial condition.

Let us present the $v(x, t, p)$ as the following:

$$v(x,t,p) = \sum_{k=0}^{\infty} v_k(x,t) p^{k\alpha} = v_0(x,t) + p^\alpha v_1(x,t) + p^{2\alpha} v_2(x,t) + ... \tag{11}$$

where $v_0(x, t,) = v(x, t, 0)$ and

$$v_k(x,t) = \frac{1}{\Gamma(1+k\alpha)} \frac{\partial^{k\alpha} v(x,t,p)}{\partial p^{k\alpha}} \Big|_{p=0}, (k \geq 1)$$

Substituting eq. (11) into eq. (10) and comparing the coefficients of each powers of p^α, that gives the following system of algebraic equation:

$$v_0(x,t) = \tilde{v} + {}_0I_x^{(3\alpha)}(u_0), v_k(x,t) = -{}_0I_x^{(3\alpha)}\left[R_k(v_{k-1}) \right], k = 2,3,..., \tag{12}$$

where

$$R_k\left[v_{k-1}(x,t)\right] = \frac{1}{\Gamma\left[1+(k-1)\alpha\right]} \frac{\partial^{(k-1)\alpha}\left[\alpha v_t^{(\alpha)} + \beta v v_x^{(\alpha)}\right]}{\partial p^{(k-1)\alpha}} \Big|_{p=0}$$

Obviously, if $v_k(x, t) = 0$, $(k \geq 1)$ then:

$$v_{k+1}(x,t) = v_{k+2}(x,t) = ... = v_{k+n}(x,t) = 0$$

Thence, we get the exact solution of eq. (7):

$$u(x,t) = v(x,t,1) = v(x,t,p) = \sum_{i=0}^{k} v_i(x,t) p^{i\alpha} = v_0(x,t) + v_1(x,t) + ... + v_k(x,t)$$

In this paper, for the sake of simplicity, we only discuss eq. (7) under the condition of $k = 1$. Then, we get the exact solution of eq. (7):

$$u(x,t) = v(x,t,1) = v_0(x,t) = \tilde{v} + {}_0I_x^{(3\alpha)} u_0 \tag{13}$$

Obviously, in using this method, how to choose $u_0 = (x, t)$, which makes $v_1(x, t) = 0$, is critical to get the exact solution of eq. (7). We shall discuss this in more detail in next section.

Two illustrative examples

To demonstrate the effectiveness of the method, three examples of local fractional Korteweg-de Vries equations with non-homogeneous term are presented.

Example 1. Consider the following local fractional KdV equation with non-homogeneous term:

$$u_t^{(\alpha)} + uu_x^{(\alpha)} + u_x^{(3\alpha)} = \frac{x^\alpha}{\Gamma(1+\alpha)}\left[E_\alpha\left(2t^\alpha\right) + E_\alpha\left(t^\alpha\right)\right] \tag{14}$$

with initial conditions:

$$u(0,t) = 0, u_x^{(\alpha)}(0,t) = E_\alpha\left(t^\alpha\right), u_x^{(2\alpha)}(0,t) = 0$$

According to the homotopy perturbation method, we can construct:

$$v_x^{(3\alpha)} = u_0(x,t) - p^\alpha\left[u_0(x,t) + v_t^{(\alpha)} + vv_x^{(\alpha)}\right] \tag{15}$$

where $u_0(x, t) = x^\alpha E_\alpha(t^\alpha)/\Gamma(1+\alpha)$ is an initial value.

Applying the inverse operator ${}_0I_x^{(3\alpha)}(\bullet)$ on both sides of eq. (15), we obtain:

$$v(x,t,p) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha\left(t^\alpha\right) + {}_0I_x^{(3\alpha)}(u_0) - p_0^\alpha I_x^{(3\alpha)}\left[u_0 + v_t^{(\alpha)} + vv_x^{(\alpha)}\right] \tag{16}$$

Substituting eq. (11) into eq. (16), collecting the same powers of p^α, and equating each coefficient od $p^{n\alpha}$ to zero yields:

$$v_0(x,t) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(t^\alpha) + {}_0I_x^{(3\alpha)}(u_0) \tag{17}$$

and

$$v_1(x,t) = -{}_0I_x^{(3\alpha)}\left[u_0 + v_{0,t}^{(\alpha)} + v_0 v_{0,t}^{(\alpha)} - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(t^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(2t^\alpha)\right]$$

$$\dots$$

$$v_n(x,t) = -{}_0I_x^{(3\alpha)}\left[v_{n-1,t}^{(\alpha)} + v_{n-1}v_{n-1,x}^{(\alpha)}\right] \tag{18}$$

Now, if we solve these eq. (18) in such a way that $v_1(x, t) = 0$,then we yield:

$$v_2(x,t) = v_3(x,t) = \dots = v_n(x,t) = 0 \tag{19}$$

Therefore, the exact solution of eq. (14) can be obtained:

$$u(x,t) = v(x,t,1) = v_0(x,t) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha\left(t^\alpha\right) + {}_0I_x^{(3\alpha)}(u_0) \tag{20}$$

We suppose:

$${}_0I_x^{(3\alpha)}(u_0) = \sum_{n=3}^{\infty} a_n(t)\frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \tag{21}$$

$$a_0(t) = 0, \quad a_1(t) = \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad a_2(t) = 0 \tag{22}$$

where $a_n(t)$, $(n \geq 3)$ are all functions to be determined.

Substituting eq. (22) into eq. (21) and then comparing the coefficient of like $x^{n\alpha}$ of the transformed equation, we can derive:

$$v_1(x,t) = -\frac{1}{\Gamma(1+4\alpha)}\left[a_1(t) - E_\alpha\left(-t^\alpha\right) - E_\alpha\left(2t^\alpha\right)\right]x^{4\alpha} - \dots. \tag{23}$$

By imposing the assumptions $v_1(x, t) = 0$, we can obtain:

$$a_3(t) = \dots = a_n(t) = 0 \tag{24}$$

Thus, the exact solution of the eq. (14):

$$u(x,t)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}\left(-t^{\alpha}\right) \tag{25}$$

Example 2. Consider the following local fractional KdV equation with non-homogeneous term, which is given in the following form:

$$u_t^{(\alpha)}+uu_x^{(\alpha)}+u_x^{(3\alpha)}=E_{\alpha}\left(2x-2t\right)^{\alpha} \tag{26}$$

with the following initial conditions:

$$u(0,t)=E_{\alpha}\left(-t^{\alpha}\right),u_x^{(\alpha)}(0,t)=E_{\alpha}\left(-t^{\alpha}\right),u_x^{(2\alpha)}(0,t)=E_{\alpha}\left(-t^{\alpha}\right)$$

According to the homotopy perturbation method, we can construct:

$$v_x^{(3\alpha)}=u_0(x,t)-p^{\alpha}\left[u_0(x,t)+v_t^{(\alpha)}+vv_x^{(\alpha)}-E_{\alpha}\left(2x-2t\right)^{\alpha}\right] \tag{27}$$

where $u_0(x, t)$ is an initial value.

Applying the inverse operator ${}_0I_x^{(3\alpha)}(\bullet)$ on both sides of eq. (27), we obtain:

$$v(x,t,p)=E_{\alpha}\left(-t^{\alpha}\right)\left[1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+{}_0I_x^{(3\alpha)}(u_0)-$$
$$-p^{\alpha}\,{}_0I_x^{(3\alpha)}\left[u_0(x,t)+v_t^{(\alpha)}+vv_x^{(\alpha)}-E_{\alpha}\left(2x-2t\right)^{\alpha}\right] \tag{28}$$

Substituting eq. (11) into eq. (28), collecting the same powers of p^{α}, and equating each coefficient of $p^{n\alpha}$ to zero yields:

$$v_0(x,t)=E_{\alpha}\left(-t^{\alpha}\right)\left[1+\frac{x^{\alpha}}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]+{}_0I_x^{(3\alpha)}(u_0) \tag{29}$$

and

$$v_1(x,t)=-{}_0I_x^{(3\alpha)}\left[u_0+v_{0,t}^{(\alpha)}+v_0v_{0,x}^{(\alpha)}-E_{\alpha}\left(2x-2t\right)^{\alpha}\right]$$
$$\dots$$
$$v_n(x,t)=-{}_0I_x^{(3\alpha)}\left(v_{n-1,t}^{(\alpha)}+v_{n-1}v_{n-1,x}^{(\alpha)}\right) \tag{30}$$

Now, if we solve eq. (30) in such a way that $v_1(x, t) = 0$, then we yield:

$$v_2(x,t)=v_3(x,t)=\dots=0 \tag{31}$$

Therefore, the exact solution of eq. (26) may be obtained:

$$u(x,t)=v_0(x,t)=u(0,t)+{}_0I_x^{(\alpha)}u_x^{(\alpha)}(0,t)+{}_0I_x^{(2\alpha)}u_x^{(\alpha)}(0,t)+{}_0I_x^{(3\alpha)}(u_0) \tag{32}$$

We suppose:

$${}_0I_x^{(3\alpha)}(u_0)=\sum_{n=3}^{\infty}a_n(t)\frac{x^{n\alpha}}{\Gamma(1+n\alpha)},\ a_0(t)=1,\ a_1(t)=\frac{x^{\alpha}}{\Gamma(1+\alpha)},\ a_2(t)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{33}$$

where $a_n(t)$, $(n \geq 3)$ are all functions to be determined.

Substituting eq. (33) and

$$E_{\alpha}\left(2x-2t\right)^{\alpha}=E_{\alpha}\left(-2t^{\alpha}\right)\sum_{n=0}^{\infty}\frac{2^{n\alpha}x^{n\alpha}}{\Gamma(1+n\alpha)}$$

into eq. (32) and then comparing the coefficient of like p^α of the transformed equation, we can deduce:

$$v_1(x,t) = -\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\left[a_0(t) + a_0^{(\alpha)}(t) + a_0(t)a_1(t) - E_\alpha\left(-2t^\alpha\right)\right] -$$
$$-\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}\left[a_1(t) + a_1^{(\alpha)}(t) + a_0(t)a_2(t) + a_1^2(t) - 2E_\alpha\left(-2t^\alpha\right)\right]$$
$$.... \tag{34}$$

where $a_n(t)$ are all functions to be determine.

By imposing the following assumptions $v_1(x, t) = 0$, we can obtain:

$$a_0(t) = a_1(t) =, ..., a_n(t) = E_\alpha\left(-t^\alpha\right) \tag{35}$$

Thus, the exact solution of the eq. (26):

$$u(x,t) = E_\alpha\left(x^\alpha\right)E_\alpha\left(-t^\alpha\right) \tag{36}$$

Conclusion

In this work, a local fractional homotopy perturbation method is introduced for solving the local fractional Korteweg-de Vries equation with non-homogeneous term in details. The test examples are showed that the suggested method can be regarded as a simple and efficient tool for computing local fractional Korteweg-de Vries equation with non-homogeneous term.

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