EXTENDING OPERATOR METHOD TO LOCAL FRACTIONAL EVOLUTION EQUATIONS IN FLUIDS

by

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This paper is aimed to solve non-linear local fractional evolution equations in fluids by extending the operator method proposed by Zenonas Navickas [15]. Firstly, we give the definitions of the generalized operator of local fractional differentiation and the multiplicative local fractional operator. Secondly, some properties of the defined operators are proved. Thirdly, a solution in the form of operator representation of a local fractional ODE is obtained by the extended operator method. Finally, with the help of the obtained solution in the form of operator representation and the travelling-wave transformations, the local fractional Kadomtsev-Petviashvili (KP) equation and the fractional Benjamin-Bona-Mahoney (BBM) equation are solved. It is shown that the extended operator method can be used for solving some other non-linear local fractional evolution equations in fluids.

Key words: *local fractional evolution equation, operator method, the generalized operator of local fractional differentiation, the multiplicative local fractional operator, the fractional KP equation, the fractional BBM equation*

Introduction

The local fractional calculus [1] developed in recent years provides a useful mathematical tool for describing the complexity and non-differentiability of real-world problems such as vibrating string, heat transfer, and fluid mechanics. It is worth mentioning that Yang *et al.* [1-9] meaningful contributions are pioneering for the sound developments of the local fractional calculus. The local fractional calculus has many graceful properties, benefiting from which some existing methods originally proposed for non-linear differential equations with integer orders, for example the variational iteration approach [10, 11], have successfully been extended to some local fractional differential equations [12-14].

In 2002, Navickas proposed the operator method [15] to represent solutions of non-linear differential equations by linear operators. For such a purpose, Navickas [15] defined the linear generalized operator and the multiplicative operator, and then proved some properties of these two operators. As far as we know this operator method has not been extended to the differential equations of fractional orders. In this paper, we shall extend the operator method to construct solutions in the form of operator representation of non-linear local fractional differential equations.

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Definitions

Definition 1. The local fractional derivative is defined [1]:

$$\mathrm{D}_\mu^{(\alpha)}\phi(\mu_0)=\left.\frac{\mathrm{d}^\alpha\phi(\mu)}{\mathrm{d}\mu^\alpha}\right|_{\mu=\mu_0}=\lim_{\mu\to\mu_0}\frac{\Delta^\alpha[\phi(\mu)-\phi(\mu_0)]}{(\mu-\mu_0)^\alpha} \tag{1}$$

where $0<\alpha\le 1$ and $\Delta^\alpha[\phi(\mu)-\phi(\mu_0)]\cong\Gamma(1+\alpha)[\phi(\mu)-\phi(\mu_0)]$ with the Euler's Gamma function:

$$\Gamma(1+\alpha)=\int_0^\infty \mu^{\alpha-1}\mathrm{e}^{-\alpha}\mathrm{d}\mu \tag{2}$$

Definition 2. Suppose $\phi(\mu)\in C_\alpha[a,b]$, then the local fractional integral of order $\alpha(0<\alpha\le 1)$ is defined [1]:

$${}_aI_b^{(\alpha)}\phi(\mu)=\frac{1}{\Gamma(1+\alpha)}\int_a^b\phi(\mu)(\mathrm{d}\mu)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta\mu_k\to 0}\sum_{k=0}^{N-1}\phi(\mu_k)(\Delta\mu_k)^\alpha \tag{3}$$

where $\Delta\mu_k=\mu_{k+1}-\mu_k$ with $\mu_0=a<\mu_1<\cdots<\mu_{N-1}=b$.

Definition 3. If we let $A=A(x,\theta,\vartheta)$ and $B=B(x,\theta,\vartheta)$ be two polynomial of variables x, θ and ϑ, then the linear operator:

$$\mathrm{D}_{\theta\vartheta}^{(\alpha)}=A\mathrm{D}_\theta^{(\alpha)}+B\mathrm{D}_\vartheta^{(\alpha)} \tag{4}$$

is called a generalized operator of local fractional differentiation.

Definition 4. The linear operator:

$$G^{(\alpha)}=G^{(\alpha)}(D_{\theta\vartheta}^{(\alpha)})=\sum_{k=0}^{+\infty}[{}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)}]^k \tag{5}$$

is called a multiplicative local fractional operator, here $({}_0I_x^{(\alpha)}D_{\theta\vartheta}^{(\alpha)})^0=1$ is the identity operator.

Properties

Properties 1. The local fractional operator of differentiation has some properties [1]:

$$\mathrm{D}_x^{(\alpha)}c=0\,,\quad \mathrm{D}_x^{(\alpha)}\frac{x^{l\alpha}}{\Gamma(1+l\alpha)}=\frac{x^{(l-1)\alpha}}{\Gamma[1+(l-1)\alpha]} \tag{6}$$

$$\mathrm{D}_x^{(\alpha)}[pf(x)+qg(x)]=p\mathrm{D}_x^{(\alpha)}f(x)+q\mathrm{D}_x^{(\alpha)}g(x) \tag{7}$$

$$\mathrm{D}_x^{(\alpha)}[f(x)g(x)]=[\mathrm{D}_x^{(\alpha)}f(x)]g(x)+f(x)[\mathrm{D}_x^{(\alpha)}g(x)] \tag{8}$$

$$\mathrm{D}_x^{(\alpha)}\frac{f(x)}{g(x)}=\frac{[\mathrm{D}_x^{(\alpha)}f(x)]g(x)-f(x)[\mathrm{D}_x^\alpha g(x)]}{g^2(x)} \tag{9}$$

$$\mathrm{D}_x^{(\alpha)}E_\alpha(hx^\alpha)=hE_\alpha(x^\alpha),\quad \mathrm{D}_x^{(\alpha)}\sin_\alpha(x^\alpha)=\cos_\alpha(x^\alpha),\quad \mathrm{D}_x^{(\alpha)}\cos_\alpha(x^\alpha)=-\sin_\alpha(x^\alpha) \tag{10}$$

where c, l, p, q, and h are all constants, and

$$E_\alpha(x^\alpha)=\sum_{k=0}^{+\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)},\quad \sin_\alpha(x^\alpha)=\sum_{k=0}^{+\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]},\quad \cos_\alpha(x^\alpha)=\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(1+2k\alpha)} \tag{11}$$

Properties 2. The local fractional operator of integration has some properties [1]:

$${}_0I_x^{(\alpha)}c=\frac{cx^\alpha}{\Gamma(1+\alpha)},\quad {}_0I_x^{(\alpha)}\frac{x^{l\alpha}}{\Gamma(1+l\alpha)}=\frac{x^{(l+1)\alpha}}{\Gamma[1+(l+1)\alpha]} \tag{12}$$

$${}_aI_b^{(\alpha)}[pf(x)+qg(x)]=p\,{}_aI_b^\alpha f(x)+q\,{}_aI_b^\alpha g(x) \tag{13}$$

$${}_aI_b^{(\alpha)}[(D_x^{(\alpha)}f(x))g(x)]=f(x)g(x)\Big|_a^b-{}_aI_b^{(\alpha)}[f(x)(D_x^{(\alpha)}g(x))] \tag{14}$$

$${}_0I_x^{(\alpha)}E_\alpha(x^\alpha)=E_\alpha(x^\alpha)-1,\ {}_0I_x^{(\alpha)}\sin_\alpha(x^\alpha)=1-\cos_\alpha(x^\alpha),\ {}_0I_x^{(\alpha)}\cos_\alpha(x^\alpha)=\sin_\alpha(x^\alpha) \tag{15}$$

Properties 3. If let $\mathrm{D}_\mu^{[(k+1)\alpha]}\phi(\mu)\in C_\alpha(a,b)$, then $\phi(\mu)$ can be expanded as [1]:

$$\phi(\mu)=\sum_{k=0}^{+\infty}\frac{\mathrm{D}_\mu^{(k\alpha)}\phi(\mu_0)}{\Gamma(1+k\alpha)}(\mu-\mu_0)^{k\alpha},\quad a<\mu_0<\mu<b \tag{16}$$

Properties 4. The generalized operator of local fractional differentiation has properties:

$$\mathrm{D}_{\theta\vartheta}^{(\alpha)}\sum_{k=1}^{n}a_kf_k=\sum_{k=0}^{n}a_k\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_k,\quad a_k\in R,\quad f_k=f_k(x,\theta,\vartheta) \tag{17}$$

$$\mathrm{D}_{\theta\vartheta}^{(\alpha)}(f_1f_2)=\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}_{\theta\vartheta}^{(k\alpha)}f_1)(\mathrm{D}_{\theta\vartheta}^{[(n-k)\alpha]}f_2),\quad n=0,1,2,\cdots \tag{18}$$

$$\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_1^{n\alpha}=f_1^{(n-1)\alpha}\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_1,\quad \mathrm{D}_{\theta\vartheta}^{(\alpha)}\frac{f_1}{f_2}=\frac{(\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_1)f_2-f_1(\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_2)}{f_2^2} \tag{19}$$

Proof. We prove the relation (18) for $n=1$, the other ones can be proved by the similar way. In view of the definition (4), we have:

$$\mathrm{D}_{\theta\vartheta}^{(\alpha)}(f_1f_2)=(A\mathrm{D}_\theta^{(\alpha)}+B\mathrm{D}_\vartheta^{(\alpha)})(f_1f_2)=A[(D_\theta^{(\alpha)}f_1)f_2+f_1D_\theta^{(\alpha)}f_2]+B[(D_\theta^{(\alpha)}f_1)f_2+f_1D_\theta^{(\alpha)}f_2]=$$

$$=(A\mathrm{D}_\theta^{(\alpha)}f_1)f_2+B(\mathrm{D}_\theta^{(\alpha)}f_1)f_2+f_1A\mathrm{D}_\theta^{(\alpha)}f_2+f_1B\mathrm{D}_\theta^{(\alpha)}f_2=(\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_1)f_2+f_1\mathrm{D}_{\theta\vartheta}^{(\alpha)}f_2 \tag{20}$$

Properties 5. The multiplicative local fractional operator has properties:

$$G^{(\alpha)}\sum_{k=1}^{n}a_kf_k=\sum_{k=0}^{+\infty}a_kG^{(\alpha)}f_k \tag{21}$$

$$G^{(\alpha)}f_1(\theta,\vartheta)=f_1(G^{(\alpha)}\theta,G^{(\alpha)}\vartheta) \tag{22}$$

$$G^{(\alpha)}\frac{f_1(\theta,\vartheta)}{f_2(\theta,\vartheta)}=\frac{f_1(G^{(\alpha)}\theta,G^{(\alpha)}\vartheta)}{f_2(G^{(\alpha)}\theta,G^{(\alpha)}\vartheta)},\quad G^{(\alpha)}(\theta^{k\alpha}\vartheta^{l\alpha})=G^{(\alpha)}(\theta^{k\alpha})G^{(\alpha)}(\vartheta^{l\alpha}) \tag{23}$$

$$G^{(\alpha)}(D_v^{(\alpha)})v^{n\alpha}=(x+v)^{n\alpha},\quad G^{(\alpha)}(D_v^{(\alpha)})f_1(v,\theta,\vartheta)=f_1(x+v,\theta,\vartheta) \tag{24}$$

Proof. We prove the relations (22) and (24). For the relation (22), we suppose that:

$$y_1=y_1(x,\theta,\vartheta)=G^{(\alpha)}\theta^\alpha,\quad y_2=y_2(x,\theta,\vartheta)=G^{(\alpha)}\vartheta^\alpha \tag{25}$$

$$z = z(x,\theta,\vartheta) = G^{(\alpha)} f_1(\theta^\alpha, \vartheta^\alpha), \quad w = w(x,\theta,\vartheta) = f_1(G^{(\alpha)}\theta^\alpha, G^{(\alpha)}\vartheta^\alpha) \tag{26}$$

then following the steps in [15] yields:

$$\begin{aligned} \mathrm{D}_x^{(\alpha)} z &= \mathrm{D}_\theta^{(\alpha)}[\sum_{k=0}^{+\infty} ({}_0I_x^{(\alpha)}\mathrm{D}_{\theta\vartheta}^{(\alpha)})^k f_1(\theta^\alpha,\vartheta^\alpha)] = \mathrm{D}_{\theta\vartheta}^{(\alpha)}[\sum_{k=0}^{+\infty} ({}_0I_x^{(\alpha)}\mathrm{D}_{\theta\vartheta}^{(\alpha)})^k f_1(\theta^\alpha,\vartheta^\alpha)] = \\ &= P\mathrm{D}_\theta^{(\alpha)} G^{(\alpha)} f_1(\theta^\alpha,\vartheta^\alpha) + Q\mathrm{D}_\vartheta^{(\alpha)} G^{(\alpha)} f_1(\theta^\alpha,\vartheta^\alpha) = P\mathrm{D}_\theta^{(\alpha)} z + Q\mathrm{D}_\vartheta^{(\alpha)} z \end{aligned} \tag{27}$$

Similarly, we have:

$$\mathrm{D}_x^{(\alpha)} y_1 = P\mathrm{D}_\theta^{(\alpha)} y_1 + Q\mathrm{D}_\vartheta^{(\alpha)} y_1, \quad \mathrm{D}_x^{(\alpha)} y_2 = P\mathrm{D}_\theta^{(\alpha)} y_2 + Q\mathrm{D}_\vartheta^{(\alpha)} y_2 \tag{28}$$

$$\begin{aligned} &\mathrm{D}_x^{(\alpha)} w = \mathrm{D}_x^{(\alpha)} f_1(y_1, y_2) = \\ &= \mathrm{D}_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} (P\mathrm{D}_\theta^{(\alpha)} y_1 + Q\mathrm{D}_\vartheta^{(\alpha)} y_1) + \mathrm{D}_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} (P\mathrm{D}_\theta^{(\alpha)} y_2 + Q\mathrm{D}_\vartheta^{(\alpha)} y_2) = \\ &= P(\mathrm{D}_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} \mathrm{D}_\theta^{(\alpha)} y_1 + \mathrm{D}_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} \mathrm{D}_\theta^{(\alpha)} y_2) + \\ &+ Q(\mathrm{D}_u^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} \mathrm{D}_\vartheta^{(\alpha)} y_1 + \mathrm{D}_v^{(\alpha)} f_1(u,v)|_{u=y_1,v=y_2} \mathrm{D}_\vartheta^{(\alpha)} y_2) = P\mathrm{D}_\theta^{(\alpha)} w + Q\mathrm{D}_\vartheta^{(\alpha)} w \end{aligned} \tag{29}$$

It is easy to see from eq. (26) that:

$$\begin{aligned} z(0,\theta,\vartheta) &= G^{(\alpha)} f_1(\theta^\alpha,\vartheta^\alpha)|_{x=0} = f_1(\theta^\alpha,\vartheta^\alpha) \\ w(0,\theta,\vartheta) &= f_1(G^{(\alpha)}\theta^\alpha, G^{(\alpha)}\vartheta^\alpha)|_{x=0} = f_1(\theta^\alpha,\vartheta^\alpha) \end{aligned} \tag{30}$$

Thus, from eqs. (27), and (29)-(30) we have $z(x,\theta,\vartheta) = w(x,\theta,\vartheta)$ which is namely the relation (22).

For the first relation in eq. (24), a direct computation shows that:

$$\begin{aligned} G^{(\alpha)}(\mathrm{D}_v^{(\alpha)}) v^{n\alpha} &= v^{n\alpha} + \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha)\Gamma[1+(n-1)\alpha]} x^\alpha v^{(n-1)\alpha} + \\ &+ \frac{\Gamma(1+n\alpha)}{\Gamma(1+2\alpha)\Gamma[1+(n-2)\alpha]} x^\alpha v^{(n-2)\alpha} + \frac{\Gamma(1+n\alpha)}{\Gamma(1+3\alpha)\Gamma[1+(n-3)\alpha]} x^\alpha v^{(n-3)\alpha} + \cdots \\ &\cdots + \frac{\Gamma(1+n\alpha)}{\Gamma[1+(n-1)\alpha]\Gamma(1+\alpha)} x^{(n-1)\alpha} v^\alpha + x^{n\alpha} = \sum_{k=0}^{+\infty} \begin{pmatrix} n\alpha \\ k\alpha \end{pmatrix} x^{k\alpha} v^{(n-k)\alpha} = (x+v)^{n\alpha} \end{aligned} \tag{31}$$

Similarly, for the second relation in eq. (24), we have:

$$\begin{aligned} G^{(\alpha)}(D_v^{(\alpha)}) f_1(v,\theta,\vartheta) &= f_1(v,\theta,\vartheta) + \frac{D_v^{(\alpha)} f_1(v,\theta,\vartheta)}{\Gamma(1+\alpha)} x^\alpha + \frac{D_v^{(2\alpha)} f_1(v,\theta,\vartheta)}{\Gamma(1+2\alpha)} x^{2\alpha} + \\ &+ \frac{D_v^{(3\alpha)} f_1(v,\theta,\vartheta)}{\Gamma(1+3\alpha)} x^{3\alpha} + \cdots = \sum_{k=0}^{+\infty} \frac{D_v^{(k\alpha)} f_1(v,\theta,\vartheta)}{\Gamma(1+k\alpha)} x^{k\alpha} = f_1(x+v,\theta,\vartheta) \end{aligned} \tag{32}$$

Properties 5. If set:

$$A = \frac{\vartheta^\alpha}{\Gamma(1+\alpha)}, \quad B = -\frac{\theta^\alpha}{\Gamma(1+\alpha)} \tag{33}$$

to eq. (4), then we have:

$$G^{(\alpha)}(\mathrm{D}_{\theta\vartheta}^{(\alpha)})\theta^{\alpha}=\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}) \tag{34}$$

$$G^{(\alpha)}(\mathrm{D}_{\theta\vartheta}^{(\alpha)})\theta^{2\alpha}=(\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}))^{2},\quad \mathrm{D}_{x}^{(\alpha)}(G^{(\alpha)}(\mathrm{D}_{\theta\vartheta}^{(\alpha)})\theta^{\alpha})=G^{(\alpha)}(\mathrm{D}_{\theta\vartheta}^{(\alpha)})\vartheta^{\alpha} \tag{35}$$

Proof. As an example, we prove the relation (34). In view of eqs. (4) and (33), we have:

$$G^{(\alpha)}(\mathrm{D}_{\theta\vartheta}^{(\alpha)})\theta^{\alpha}=\theta^{\alpha}+\vartheta^{\alpha}\frac{x^{\alpha}}{\Gamma(1+\alpha)}-\theta^{\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}-\vartheta^{\alpha}\frac{x^{3\alpha}}{\Gamma(1+3\alpha)}+\theta^{\alpha}\frac{x^{4\alpha}}{\Gamma(1+4\alpha)}+$$
$$\vartheta^{\alpha}\frac{x^{5\alpha}}{\Gamma(1+5\alpha)}-\cdots=\theta^{\alpha}\cos_{\alpha}(x^{\alpha})+\vartheta^{\alpha}\sin_{\alpha}(x^{\alpha}) \tag{36}$$

Properties 6. If let:

$$\mathrm{D}_{v\theta\vartheta}^{(\alpha)}=\mathrm{D}_{v}^{(\alpha)}+A\mathrm{D}_{\theta}^{(\alpha)}+B\mathrm{D}_{\vartheta}^{(\alpha)} \tag{37}$$

then we have:

$$G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})f_1(v)=G^{(\alpha)}(\mathrm{D}_{v}^{(\alpha)})f_1(v)=f_1(x+v) \tag{38}$$

$$G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})f_1(v,\theta,\vartheta)=f_1(G^{(\alpha)}(\mathrm{D}_{v}^{(\alpha)})v,G^{(\alpha)}(\mathrm{D}_{v}^{(\alpha)})\theta,G^{(\alpha)}(\mathrm{D}_{v}^{(\alpha)})\vartheta) \tag{39}$$

Proof. We can see that the relation (38) is obvious. The proof of the relation (39) is similar to that of the relation (22) and we omit it here for simplification.

Theorem

Theorem 1. If a local fractional ODE is given by:

$$\mathrm{D}_{x}^{(2\alpha)}u=P(x,u,\mathrm{D}_{x}^{(\alpha)}u),\quad u=u(x,\theta,\vartheta),\quad u(v,\theta,\vartheta)=\theta^{\alpha},\quad \mathrm{D}_{x}^{(2\alpha)}u\,|_{x=v}=\vartheta^{\alpha} \tag{40}$$

where $P(x,\theta,\vartheta)$ is a polynomial of variables x, θ, and ϑ, then eq. (40) has a solution of the operator representation:

$$u=\sum_{k=0}^{+\infty}\frac{(x-v)^{k\alpha}}{\Gamma(1+\alpha)}(\mathrm{D}_{v}^{(\alpha)}+\vartheta^{\alpha}\mathrm{D}_{\theta}^{(\alpha)}-P(v,\theta,\vartheta)\mathrm{D}_{\vartheta}^{(\alpha)})^{k}\theta^{\alpha},\quad v\in R \tag{41}$$

Proof. Let

$$\mathrm{D}_{v\theta\vartheta}^{(\alpha)}=\frac{\mathrm{D}_{v}^{(\alpha)}+\theta^{\alpha}\mathrm{D}_{\theta}^{(\alpha)}+P(v,\theta,\vartheta)\mathrm{D}_{\vartheta}^{(\alpha)}}{\Gamma(1+\alpha)},\quad z(x,\theta,\vartheta,v)=G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\theta^{\alpha} \tag{42}$$

we have:

$$\mathrm{D}_{x}^{(\alpha)}z(x,\theta,\vartheta,v)=\mathrm{D}_{x}^{(\alpha)}G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\theta^{\alpha}=\mathrm{D}_{v\theta\vartheta}^{(\alpha)}G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\theta^{\alpha}=$$
$$=G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\mathrm{D}_{v\theta\vartheta}^{(\alpha)}\theta^{\alpha}=G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\vartheta^{\alpha} \tag{43}$$

$$D_{x}^{(2\alpha)}z(x,\theta,\vartheta,v)=D_{x}^{(\alpha)}G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\vartheta^{\alpha}=G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})P(v,\theta,\vartheta)=$$
$$=P(x+`v,G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\theta^{\alpha},G^{(\alpha)}(D_{v\theta\vartheta}^{(\alpha)})\vartheta^{\alpha}) \tag{44}$$

and then arrive at:

$$\mathrm{D}_x^{(2\alpha)} z(x-v,\theta,\vartheta,v) = P[x, z(x-v,\theta,\vartheta,v), \mathrm{D}_x^{(\alpha)} z(x-v,\theta,\vartheta,v)] \tag{45}$$

Thus, from eqs. (17) and (45) we obtain:

$$u = z(x-v,\theta,\vartheta,v) = G^{(\alpha)}(\mathrm{D}_{v\theta\vartheta}^{(\alpha)})\theta^\alpha \,|_{x=x-v} = \sum_{k=0}^{+\infty} \frac{(x-v)^{k\alpha}}{\Gamma(1+\alpha)} [\mathrm{D}_v^{(\alpha)} + \vartheta^\alpha \mathrm{D}_\theta^{(\alpha)} + P(v,\theta,\vartheta)\mathrm{D}_\vartheta^{(\alpha)}]^k \theta^\alpha \tag{46}$$

Applications

Example 1. Application to the fractional KP equation:

$$\mathrm{D}_x^{(\alpha)}(\mathrm{D}_t^{(\alpha)} u + 6u\mathrm{D}_x^{(\alpha)} + \mathrm{D}_x^{(3\alpha)} u) + \mathrm{D}_y^{(\alpha)} u = 0 \tag{47}$$

Firstly, we take the travelling-wave transformation:

$$\xi = x + y + t \tag{48}$$

then eq. (47) is reduced into:

$$\mathrm{D}_\xi^{(\alpha)}(\mathrm{D}_\xi^{(\alpha)} u + 6u\mathrm{D}_\xi^{(\alpha)} u + \mathrm{D}_\xi^{(3\alpha)} u) + \mathrm{D}_\xi^{(2\alpha)} u = 0 \tag{49}$$

Secondly, we integrate eq. (49) with respective to ξ twice, then eq. (49) becomes:

$$2u + 3u^2 + \mathrm{D}_\xi^{(2\alpha)} u = 0 \tag{50}$$

Finally, we suppose:

$$P(\xi, u, \mathrm{D}_\xi^{(2\alpha)} u) = -2u - 3u^2, \quad u(v) = \theta^\alpha, \quad \mathrm{D}_\xi^{(\alpha)} u(\xi)\,|_{\xi=v} = \vartheta^\alpha \tag{51}$$

and then obtain a solution of operator representation by using *Theorem 1*:

$$u = \sum_{k=0}^{+\infty} \frac{(\xi-v)^{k\alpha}}{\Gamma(1+\alpha)} [\mathrm{D}_v^{(\alpha)} + \vartheta^\alpha \mathrm{D}_\theta^{(\alpha)} + (3\theta^{2\alpha} + 2\theta^\alpha)\mathrm{D}_\vartheta^{(\alpha)}]^k \theta^\alpha \tag{52}$$

Example 2. Application to the fractional BBM equation:

$$\mathrm{D}_t^{(\alpha)} u + 2\omega u \mathrm{D}_x^{(\alpha)} u + 3u\mathrm{D}_x^{(\alpha)} u - \mathrm{D}_{xxt}^{(3\alpha)} u = 0, \quad \omega = \text{const.} \tag{53}$$

Similarly, we take the travelling-wave transformation:

$$\xi = x + t \tag{54}$$

then eq. (53) is converted into:

$$\mathrm{D}_\xi^{(\alpha)} u + 2\omega u \mathrm{D}_\xi^{(\alpha)} u + 3u\mathrm{D}_\xi^{(\alpha)} u - \mathrm{D}_{\xi\xi\xi}^{(3\alpha)} u = 0 \tag{55}$$

Integrating eq. (55) with respective to ξ once, we have:

$$(1+2\omega)u + \frac{3}{2}u^2 - \mathrm{D}_\xi^{(2\alpha)} u = 0 \tag{56}$$

Supposing

$$P(\xi, u, \mathrm{D}_\xi^{(2\alpha)} u) = (1+2\omega)u + \frac{3}{2}u^2, \quad u(v) = \theta^\alpha, \quad \mathrm{D}_\xi^{(\alpha)} u(\xi)\,|_{\xi=v} = \vartheta^\alpha \tag{57}$$

and using *Theorem 1*, then we obtain a solution of operator representation:

$$u=\sum_{k=0}^{+\infty}\frac{(\xi-\nu)^{k\alpha}}{\Gamma(1+\alpha)}\{\mathrm{D}_{\nu}^{(\alpha)}+\vartheta^{\alpha}\mathrm{D}_{\theta}^{(\alpha)}-[(1+2\omega)\theta+\frac{3}{2}\theta^{2}]\mathrm{D}_{\vartheta}^{(\alpha)}\}^{k}\theta^{\alpha} \tag{58}$$

Conclusion

In summary, the extended operator method has been established for constructing solutions of operator representation of non-linear local fractional evolution equations in fluids. When the fractional-order tends to 1, the extended operator method, the generalized operator of local fractional differentiation (4), the multiplicative local fractional operator (5), the properties (17)-(19), (21)-(24), and (34)-(35), and the obtained solution (46) degenerates into those of Navickas' [15]. To the best of our knowledge, solutions (52) and (58) of the local fractional KP and BBM equations are novel.

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Nomenclature

a, b – real numbers, [–]
$\mathrm{d}^{\alpha}/\mathrm{d}^{\alpha}x$ – first local fractional derivative [–]
t – time co-ordinate, [s]

Greek symbols

α – fractional order, [–]
x, y – space co-ordinates, [m]
ν, μ, θ, ϑ – space co-ordinates, [m]

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