

A NEW APPROXIMATE ANALYTICAL METHOD FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

by

Shao-Wen YAO and Kang-Le WANG*

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China

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In this paper, a new approximate analytical method is established, and it is useful in constructing approximate analytical solution a system of fractional differential equations. The results show that our method is reliable and efficient for solving the fractional system.

Key words: *Caputo fractional derivative, analytical solution, fractional differential equations, Daftardar-Gejji and Jafari method*

Introduction

In 2006, Daftardar-Gejji and Jafari (DGJ) method (DGJM) was proposed by Daftardar-Gejji and Jafari [1]. This method is now widely used by many researchers to study fractional PDE. It was shown that the method is more powerful than existing techniques such as the Adomian method [2, 3], travelling-wave method [4-8], and homotopy analysis method (HAM) [9]. The method gives rapidly convergent successive approximations of the exact solution, and it has no specific requirements [10, 11], such as small parameters, linearization, Adomian polynomials for non-linear terms, *etc.* Recently, Daftardar-Gejji *et al.* [12] and Yang [13] have found the exact solution and approximate solution of fractional differential equations by using DGJ method.

In this paper, we have applied DGJM to study the following systems of fractional partial differential equations:

$$\begin{cases} D_t^\alpha u(x,t) = f(u,v,u_x,v_x) \\ D_t^\beta v(x,t) = g(u,v,u_x,v_x) \end{cases} \quad (1)$$

with the initial conditions:

$$u(x,0) = \varphi(x), v(x,0) = \psi(x) \quad (2)$$

where the fractional derivative is understood in the Caputo sense [14], α and β are parameters describing the order of the Caputo fractional derivative ($0 < \alpha \leq 1$, $0 < \beta \leq 1$), and f , g , φ , and ψ are given functions.

Such systems arise in various areas, especially in the study of chemical reactions, in population dynamics and in mathematical biology [15-21].

In [14], Jafari and Seifi studied a special case of the above systems:

$$\begin{cases} D_t^\alpha u(x,t) = v_x - v - u \\ D_t^\beta v(x,t) = u_x - v - u \end{cases} \quad (3)$$

* Corresponding author, e-mail: kangle83718@163.com

with the initial conditions as:

$$u(x, 0) = sh(x), \quad v(x, 0) = ch(x) \quad (4)$$

From [14], we can see that the structure of the solutions to the systems of fractional differential equations are very different from those of integer-order differential equations.

The aim of the present work is to construct approximate solution the problem of eqs. (1) and (2) by using DGJM. Our results show that the method introduces a reliable and efficient process for solving the systems.

Basic definitions of fractional calculus

In this section, we give some basic definitions and properties of fractional calculus theory which shall be used in this paper [14].

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ , $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

The Riemann-Liouville fractional integral operator is defined:

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds, \quad J^0 f(x) = f(x)$$

The properties of the operator J^α can be found in and we mention only the following [14]: For $\alpha, \beta \geq 0$, $x > 0$, and $\gamma > -1$:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha (x^\gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha+\gamma)} x^{\gamma+\alpha}$$

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined [14]:

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds$$

for $m-1 < \alpha \leq m$, $m \in N^+$, $x > 0$, and $f \in C_{-1}^m$.

We recall here two of its basic properties [14-21]:

$$D^\alpha J^\alpha f(x) = f(x)$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0$$

The DGJ method

In this section, we illustrate the basic idea of DGJM [1]. We consider the following general function equation:

$$u = L(u) + N(u) + f \quad (5)$$

where L is a linear operator, N is a non-linear operator from a Banach space $B \rightarrow B$ and f a known function. We are looking for a solution u of eq. (9) having the series form:

$$u = \sum_{i=0}^{\infty} u_i \quad (6)$$

The non-linear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \quad (7)$$

From eqs. (6) and (7), eq. (5) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right) \right\} \quad (8)$$

Define the recurrence relation:

$$u_0 = f(x), \quad u_1 = L(u_0) + H_0, \quad u_m = L(u_m) + H_m, \quad m = 1, 2, \dots \quad (9)$$

where

$$H_0 = N(u_0)$$

$$H_m = N\left(\sum_{i=0}^m u_i\right) - N\left(\sum_{i=0}^{m-1} u_i\right), \quad m = 1, 2, \dots$$

Then k -term approximate solution of eq. (5) is given by:

$$u = u_0 + u_1 + \dots + u_{k-1} \quad (10)$$

Solution for the system of eqs. (1) and (2)

In this section we derive an algorithm of the DGJM for solving the systems of eqs. (1) and (2).

To use DGJM, we rewrite the systems of eqs. (1) and (2) as:

$$\begin{cases} D_t^\alpha u(x, t) = h_1(x) + L_1(X) + N_1(X) \\ D_t^\beta v(x, t) = h_2(x) + L_2(X) + N_2(X) \end{cases} \quad (11)$$

with the initial conditions:

$$u(x, 0) = \varphi(x), \quad v(x, 0) = \psi(x) \quad (12)$$

where h_1 and h_2 are known functions, L_1 and L_2 are linear operators, N_1 and N_2 are non-linear operators, and $X = (u, v, u_x, v_x)$.

From the basic properties of the operators J^α and D^α , the systems of eqs. (11) and (12) are equivalent to:

$$\begin{cases} u(x, t) = J_t^\alpha [h_1(x) + L_1(X) + N_1(X)] + \varphi(x) \\ v(x, t) = J_t^\beta [h_2(x) + L_2(X) + N_2(X)] + \psi(x) \end{cases} \quad (13)$$

Suppose that the solution of eq. (13) takes the form:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t), \quad v(x, t) = \sum_{k=0}^{\infty} v_k(x, t)$$

Let:

$$G_0 = N_1(X_0), \quad D_0 = N_2(X_0)$$

$$G_1 = N_1(X_0 + X_1) - N_1(X_0)$$

$$D_1 = N_2(X_0 + X_1) - N_2(X_0)$$

$$G_m = N_1(X_0 + X_1 + \dots + X_m) - N_1(X_0 + X_1 + \dots + X_{m-1})$$

$$D_m = N_2(X_0 + X_1 + \dots + X_m) - N_2(X_0 + X_1 + \dots + X_{m-1})$$

where $m = 1, 2, \dots$, and $X_k = (u_k, v_k, u_{kx}, v_{kx})$, $k = 0, 1, \dots$

Then we can construct the recurrence relation:

$$\begin{aligned} u_0(x, t) &= \varphi(x), \quad v_0(x, t) = \psi(x) \\ u_1(x, t) &= J_t^\alpha [h_1(x) + L_1(X_0) + G_0(X_0)] \\ v_1(x, t) &= J_t^\beta [h_2(x) + L_2(X_0) + D_0(X_0)] \\ u_2(x, t) &= J_t^\alpha [L_1(X_1) + G_1(X_0, X_1)] \\ v_2(x, t) &= J_t^\beta [L_2(X_1) + D_1(X_0, X_1)] \\ u_3(x, t) &= J_t^\alpha [L_1(X_2) + G_2(X_0, X_1, X_3)] \\ v_3(x, t) &= J_t^\beta [(L_2(X_2) + D_2(X_0, X_1, X_2))] \end{aligned} \quad (14)$$

and so on.

Thus the solutions in the series form are given by:

$$\begin{aligned} u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots, \\ v(x, t) &= v_0(x, t) + v_1(x, t) + v_2(x, t) + \dots \end{aligned}$$

To give a clear over of the algorithm previously introduced, two illustrative system of PDE, a linear and another non-linear, have been selected to demonstrate the reliability and efficiency of the method.

Example 1. Consider the following linear system [16]:

$$\begin{cases} D_t^\alpha u(x, t) = v_x - v - u \\ D_t^\beta v(x, t) = u_x - v - u \end{cases}$$

with the initial conditions as:

$$u(x, 0) = sh(x), \quad v(x, 0) = ch(x)$$

The initial value problem are equivalent to the following fractional integral equations:

$$\begin{cases} u(x, t) = J_t^\alpha (v_x - v - u) + sh(x) \\ v(x, t) = J_t^\beta (u_x - v - u) + ch(x) \end{cases}$$

Following the algorithm given in eq. (14):

$$\begin{aligned} u_0(x, t) &= sh(x), \quad v_0(x, t) = ch(x) \\ u_1(x, t) &= J_t^\alpha (v_{0x} - v_0 - u_0) = -\frac{ch(x)}{\Gamma(1+\alpha)} t^\alpha \\ v_1(x, t) &= J_t^\beta (u_{0x} - v_0 - u_0) = -\frac{sh(x)}{\Gamma(1+\beta)} t^\beta \\ u_2(x, t) &= J_t^\alpha (v_{1x} - v_1 - u_1) = \frac{ch(x)}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{sh(x) - ch(x)}{\Gamma(1+\alpha+\beta)} t^{\alpha+\beta} \\ v_2(x, t) &= J_t^\beta (u_{1x} - v_1 - u_1) = \frac{sh(x)}{\Gamma(1+2\beta)} t^{2\beta} + \frac{ch(x) - sh(x)}{\Gamma(1+\alpha+\beta)} t^{\alpha+\beta} \end{aligned}$$

and so on.

Hence:

$$u(x,t) = sh(x) - \frac{ch(x)}{\Gamma(1+\alpha)}t^\alpha + \frac{ch(x)}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{sh(x)-ch(x)}{\Gamma(1+\alpha+\beta)}t^{\alpha+\beta} + \dots,$$

$$v(x,t) = ch(x) - \frac{sh(x)}{\Gamma(1+\beta)}t^\beta + \frac{sh(x)}{\Gamma(1+2\beta)}t^{2\beta} + \frac{ch(x)-sh(x)}{\Gamma(1+\alpha+\beta)}t^{\alpha+\beta} + \dots$$

If $\alpha = \beta = 1$, then:

$$u(x,t) = sh(x) \left(1 + \frac{t^2}{2!} + \dots \right) - ch(x) \left(t + \frac{t^3}{3!} + \dots \right)$$

$$v(x,t) = ch(x) \left(1 + \frac{t^2}{2!} + \dots \right) - sh(x) \left(t + \frac{t^3}{3!} + \dots \right)$$

which are exactly the same as the solutions obtained in [21] converging to the exact solutions, $u(x, t) = sh(x - t)$, $v(x, t) = ch(x - t)$.

Example 2. Consider the following non-linear system:

$$\begin{cases} D_t^\alpha u(x,t) = 1 - u - u_x v \\ D_t^\beta v(x,t) = 1 + v + u_x v_x \end{cases}$$

with the initial conditions as: $u(x, 0) = e^x$, $v(x, 0) = e^{-x}$

Here $L_1 = -u$, $L_2 = v$, $N_1 = -u_x v$, and $N_2 = u v_x$.

Following eq. (18), we obtain:

$$u_0(x,t) = e^x, \quad v_0(x,t) = e^{-x}, \quad u_1(x,t) = \frac{-e^x t^\alpha}{\Gamma(1+\alpha)}, \quad v_1(x,t) = \frac{e^{-x} t^\beta}{\Gamma(1+\beta)}$$

$$u_2(x,t) = \frac{(e^x + 1)t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{\Gamma(1+\alpha+\beta)t^{2\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)}$$

$$v_2(x,t) = \frac{(e^{-x} - 1)t^{2\beta}}{\Gamma(1+2\beta)} + \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{\Gamma(1+\alpha+\beta)t^{\alpha+2\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)}$$

Thus we find that the solutions are:

$$u(x,t) = e^x - \frac{e^x t^\alpha}{\Gamma(1+\alpha)} + \frac{(e^x + 1)t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{\Gamma(1+\alpha+\beta)t^{2\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)} + \dots,$$

$$v(x,t) = e^{-x} + \frac{e^{-x} t^\beta}{\Gamma(1+\beta)} + \frac{(e^{-x} - 1)t^{2\beta}}{\Gamma(1+2\beta)} + \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{\Gamma(1+\alpha+\beta)t^{\alpha+2\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)} + \dots$$

If $\alpha = \beta = 1$ we obtain the close-form solutions [21]:

$$u(x,t) = e^{x-t}, \quad v(x,t) = e^{-x+t}$$

Conclusion

In this work, a new approximate analytical method was applied to handle the system of the fractional differential equations within the Caputo fractional derivative, The results show that it is more reliable, efficient and accurate than the other ones.

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