APPROXIMATE ANALYTIC SOLUTIONS OF MULTI-DIMENSIONAL FRACTIONAL HEAT-LIKE MODELS WITH VARIABLE COEFFICIENTS

by

Jianshe SUN [a,b]

[a] School of Mathematics, China University of Mining and Technology, Xuzhou, China
[b] School of Mathematics, Jiaozuo Teacher's College, Jiaozuo, China

Original scientific paper
https://doi.org/10.2298/TSCI180612256S

In this work, the fractional power series method is applied to solve the 2-D and 3-D fractional heat-like models with variable coefficients. The fractional derivatives are described in the Liouville-Caputo sense. The analytical approximate solutions and exact solutions for the 2-D and 3-D fractional heat-like models with variable coefficients are obtained. It is shown that the proposed method provides a very effective, convenient and powerful mathematical tool for solving fractional differential equations in mathematical physics.

Key words: *heat-like models fractional power series, fractional differential equation with variable coefficients, Liouville-Caputo fractional derivative*

Introduction

In this paper, we consider the 3-D fractional order heat-like model:

$$\mathrm{D}_t^{\alpha} u = f(x,y,z)\mathrm{D}_x^{\beta_1} u + g(x,y,z)\mathrm{D}_y^{\beta_2} u + h(x,y,z)\mathrm{D}_z^{\beta_3} u \tag{1}$$

with the initial condition:

$$u(x,y,z,0) = \mu_1(x,y,z) \tag{2}$$

and the 2-D order heat-like model:

$$\mathrm{D}_t^{\alpha} u = f(x,y)\mathrm{D}_x^{\beta_1} u + g(x,y)\mathrm{D}_y^{\beta_2} u \tag{3}$$

with the initial condition:

$$u(x,y,0) = \mu_2(x,y) \tag{4}$$

where $u = u(x,y,z,t)$, $0 < \alpha \le 1$, $\mathrm{D}_t^{\alpha} u(x,y,z,t)$ is the Liouville-Caputo fractional derivative [1], $f(x,y,z)$, $g(x,y,z)$, and $h(x,y,z)$ are any functions with respect to the variables x, y, and z. In the case of $\beta_j = 2$ $(j = 1,2,3)$, then eq. (1) reduces to a fractional heat-like equation with variable coefficients [2].

The fractional power series method (FPSM) have played an important role in solving the fractional differential equations in applied and engineering sciences [3-6]. The FPSM was proposed to solve the fractional diffusion equation within Caputo fractional derivative

Author's e-mail: sunjianshe@126.com

[7] and was used to solve the fractional 1-D heat-like equations with variable coefficients [8]. The target of the paper is to solve the 2-D and 3-D fractional heat-like models with variable coefficients.

The FPS and FPSM

The basic idea of the FPS

A power series of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots, \tag{5}$$

is called a FPS about t_0, where $0 \le m-1 < \alpha \le m$, $m \in N^+$, t $(t \ge t_0)$ is a variable and c_n are the coefficients of the series.

Let the FPS $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ be the radius of convergence, denoted as $r > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{na}$ by on $0 \le t < r$, then for $m-1 < \alpha \le m$ and $0 \le t < r$, we have:

$$\mathrm{D}^{\alpha} f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} t^{(n-1)\alpha} \tag{6}$$

For more information for the FPS, [3-8].

Applications

The FPSM for solving the 3-D fractional heat-like model with variable coefficients

Suppose that the solution of eqs. (1) and (2) takes the form:

$$u(x,y,z,t) = \sum_{k=0}^{\infty} a_k (x,y,z) t^{\alpha k} \tag{7}$$

where $a_k(x,y,z)$, $(k = 1,2,\cdots)$, is denoted as the components of the function $u(x,y,z,t)$, which will be determined recursively.

Making use of eq. (2), one obtains:

$$a_0(x,y,z) = \mu_1(x,y,z) \tag{8}$$

From eq. (6), one gets:

$$\mathrm{D}_t^{\alpha} u(x,y,z,t) = \sum_{k=1}^{\infty} \frac{a_k(x,y,z)\Gamma(\alpha k+1)}{\Gamma[\alpha(k-1)+1]} t^{\alpha(k-1)} \tag{9}$$

From eq. (7), it is easy to see that:

$$\mathrm{D}_x^{\beta_1} u = \mathrm{D}_x^{\beta_1} a_0(x,y,z) + t^{\alpha} \mathrm{D}_x^{\beta_1} a_1(x,y,z) + t^{2\alpha} \mathrm{D}_x^{\beta_1} a_2(x,y,z) + \cdots \tag{10}$$

$$\mathrm{D}_y^{\beta_2} u = \mathrm{D}_y^{\beta_2} a_0(x,y,z) + t^{\alpha} \mathrm{D}_y^{\beta_2} a_1(x,y,z) + t^{2\alpha} \mathrm{D}_y^{\beta_2} a_2(x,y,z) + \cdots \tag{11}$$

and

$$\mathrm{D}_z^{\beta_3} u = \mathrm{D}_z^{\beta_3} a_0(x,y,z) + t^{\alpha} \mathrm{D}_z^{\beta_3} a_1(x,y,z) + t^{2\alpha} \mathrm{D}_z^{\beta_3} a_2(x,y,z) + \cdots \tag{12}$$

Substituting eqs. (9)-(12) into eq. (1), we have:

$$\sum_{k=1}^{\infty} a_k(x,y,z)\frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]}t^{(k-1)\alpha} = f(x,y,z)\sum_{k=0}^{\infty} t^{k\alpha}\mathrm{D}_x^{\beta_1} a_k(x,y,z) +$$
$$+g(x,y,z)\sum_{k=0}^{\infty} t^{k\alpha}\mathrm{D}_y^{\beta_2} a_k(x,y,z) + h(x,y,z)\sum_{k=0}^{\infty} t^{k\alpha}\mathrm{D}_z^{\beta_3} a_k(x,y,z) \tag{13}$$

Comparing the coefficients of $t^{k\alpha}$ in eq. (13), we have:

$$a_k(x,y,z) = \frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)}(f\mathrm{D}_x^{\beta_1} a_{k-1} + g\mathrm{D}_y^{\beta_2} a_{k-1} + h\mathrm{D}_z^{\beta_3} a_{k-1}) \tag{14}$$

such that:

$$u(x,y,z,t) = \sum_{k=0}^{\infty} t^{k\alpha}\frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)}(f\mathrm{D}_x^{\beta_1} a_{k-1} + g\mathrm{D}_y^{\beta_2} a_{k-1} + h\mathrm{D}_z^{\beta_3} a_{k-1}) \tag{15}$$

where $k = 1, 2, \cdots$.

The FPSM for solving the 2-D fractional heat-like model with variable coefficients

Let us consider the 2-D fractional heat-like model with variable coefficients:

$$\mathrm{D}_t^{\alpha} u = \frac{1}{2}(y^2\mathrm{D}_x{}^2 u + x^2\mathrm{D}_y^2 u), \quad 0 < x, \quad y < 1, \quad t > 0 \tag{16}$$

subject to the initial condition:

$$u(x,y,0) = y^2 \tag{17}$$

In this case, we can write the solutions of eqs. (16) and (17) as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} a_k(x,y)t^{k\alpha} \tag{18}$$

Obviously,

$$a_0(x,y) = y^2 \tag{19}$$

From eq. (6) we present:

$$\mathrm{D}_t^{\alpha} u = \sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{\Gamma[(k-1)\alpha+1]}a_k(x,y)t^{(k-1)\alpha} =$$
$$= \Gamma(\alpha+1)a_1(x,y) + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}a_2(x,y)t^{\alpha} + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}a_3(x,y)t^{2\alpha} + \cdots \tag{20}$$

With the aid of eq. (18), it is easy to see that:

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty}\frac{\partial^2 a_k}{\partial x^2}t^{k\alpha} = \frac{\partial^2 a_0}{\partial x^2} + \frac{\partial^2 a_1}{\partial x^2}t^{\alpha} + \frac{\partial^2 a_2}{\partial x^2}t^{2\alpha} + \cdots \tag{21}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{k=0}^{\infty}\frac{\partial^2 a_k}{\partial y^2}t^{k\alpha} = \frac{\partial^2 a_0}{\partial y^2} + \frac{\partial^2 a_1}{\partial y^2}t^{\alpha} + \frac{\partial^2 a_2}{\partial y^2}t^{2\alpha} + \cdots \tag{22}$$

Substituting the expansion of eqs. (20)-(22) into eq. (16), it follows that:

$$\Gamma(\alpha+1)a_1(x,y)+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}a_2(x,y)t^{\alpha}+\frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}a_3(x,y)t^{2\alpha}+\cdots=$$

$$=\frac{1}{2}\left(y^2\frac{\partial^2 a_0}{\partial x^2}+x^2\frac{\partial^2 a_0}{\partial y^2}\right)+\frac{1}{2}\left(y^2\frac{\partial^2 a_1}{\partial x^2}+x^2\frac{\partial^2 a_1}{\partial y^2}\right)t^{\alpha}+\frac{1}{2}\left(y^2\frac{\partial^2 a_2}{\partial x^2}+x^2\frac{\partial^2 a_2}{\partial y^2}\right)t^{2\alpha}+\cdots \quad (23)$$

Comparing the coefficients of eqs. (20) and (23), we have:

$$a_k(x,y)=\frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)}\left(\frac{1}{2}y^2\frac{\partial^2 a_{k-1}}{\partial x^2}+\frac{1}{2}x^2\frac{\partial^2 a_{k-1}}{\partial y^2}\right),\quad (k=1,2,\cdots) \quad (24)$$

Substituting eq. (19) into eq. (24), we present:

$$a_1(x,y)=\frac{x^2}{\Gamma(\alpha+1)},\quad a_2(x,y)=\frac{y^2}{\Gamma(2\alpha+1)},\quad a_3(x,y)=\frac{x^2}{\Gamma(3\alpha+1)}, \quad (25)$$

$$a_4(x,y)=\frac{y^2}{\Gamma(4\alpha+1)}, \quad (26)$$

and so on.

Therefore, we obtain:

$$u(x,y,t)=y^2+\frac{x^2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{y^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^2t^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{y^2t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots. \quad (27)$$

If $\alpha=1$, then we have the exact solution:

$$u(x,y,t)=y^2+x^2t+\frac{y^2t^2}{2!}+\frac{x^2t^3}{3!}+\frac{y^2t^4}{4!}+\cdots=x^2\sinh t+y^2\cosh t \quad (28)$$

Conclusion

In the present task, the FPSM has been successfully applied to solve 2-D and 3-D fractional heat-like models with variable coefficients. It is shown that the FPSM is a simple and effective method for solving exact approximate solutions of fractional partial differential equations with variable coefficients.

Acknowledgment

This work is supported by the key scientific research project of Henan higher education institutions (Grant No.20B110009) and the soft science research project of Henan province (Grant No.182400410540).

Nomenclature

k – natural number, [–]
N – positive integer, [–]
x, y, z – space co-ordinates, [m]
t – time, [s]

Greek symbols

α – fractional order, [–]
β_i – fractional order, [–]

References

[1] Yang, X. J., *General Fractional Derivatives: Theory, Methods and Applications*, CRC Press, Boka Raton, Fla., USA, 2019

[2] Bhrawy, A. H., A New Spectral Algorithm for Time-Space Fractional Partial Differential Equations with Subdiffusion and Superdiffusion, *Proceedings of the Romanian Academy Series A*, *17* (2016), 3, pp. 39-47
[3] Yang, A. M., *et al.,* Local Fractional Series Expansion Method for Solving Wave and Diffusion Equations on Cantor Sets, *Abstract and Applied Analysis*, *2013* (2013), ID 3510572
[4] Yang, X. J., *et al., Local Fractional Integral Transforms and their Applications*, Academic Press, New York, USA, 2015
[5] Jafari, H., *et al.,* Local Fractional Series Expansion Method for Solving Laplace and Schrodinger Equations on Cantor Sets within Local Fractional Operators, *International Journal of Mathematics and Computer Research*, *2* (2014), 11, pp. 736-744
[6] Kumar, A., *et al.,* Residual Power Series Method for Fractional Diffusion Equations, *Fundamenta Informaticae*, *151* (2017), 1-4, pp.213-230
[7] Yan, S. P., *et al.,* A Novel Series Method for Fractional Diffusion Equation within Caputo Fractional Derivative, *Thermal Science, 20* (2016), 3, pp. S695-S699
[8] Cui, R. Q., *et al.,* An Analytical Solution for a Fractional Heat-Like Equation with Variable Coefficients, *Thermal Science, 21* (2017), 4, pp. 1759-1764

Paper submitted: June 12, 2018
Paper revised: December 20, 2018
Paper accepted: Janary 25, 2019