# APPLICATIONS OF THE FOURIER-LIKE INTEGRAL TRANSFORM IN THE WAVE AND HEAT TRANSFER PROBLEMS

by

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*In this article, some new properties of a novel integral transform termed the Fourier-Yang are explored. The Fourier-Yang integral transforms of several basic functions are given firstly. With the aid of the new integral transform, a 1-D wave equation and 2-D heat transfer equation are solved. The results show that the Fourier-Yang integral transform is efficient in solving PDE.*

Key words: *Fourier-like integral transform, analytical solution, heat transfer equation, wave equation*

## Introduction

Integral transforms have been applied to solving the key issues involving mechanics, chemistry, physics, thermal science, and interdisciplinary areas [1, 2]. For example, the Laplace integral transform plays the important role in transient thermal stresses [3], fluid mechanics [4], and viscoelastic fluids [5]. The Fourier integral transform has become the powerful tool in solving the volume integral equations and the Cauchy integral equation [6-8], and the Sumudu integral transform was utilized to solve PDE in [9-11].With the development of the integral transform, some new integral transforms, such as the Elzaki transform [12, 13], the Fourier-Yang transform [14, 15], and the Laplace-Carson transform [16], were also suggested to solve more differential equations. Recently, a new Fourier-like integral transform adopted to deal with a steady heat transfer problem is given [17]. However, the properties of the new integral transform are incomplete, and it has not been employed to solve the wave and the 2-D heat transfer equations.

This paper aims to extend some new properties of the Fourier-Yang integral transform and give the integral transform of some basic functions. Moreover, the PDE proposed in the 1-D wave and the 2-D heat transfer problems are solved by the technique of the integral transform for the first time.

## The Fourier-Yang integral transfor

In this section, the definitions of the Fourier integral transform and the Fourier-Yang integral transform are recalled, some new properties of the Fourier-Yang integral transform are given firstly. In addition, the integral transforms of some functions are defined.

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The Fourier integral transform of the function $\varphi(t)$ is denoted as [18]:

$$\Phi(\omega) = F\left[\varphi(t)\right] = \int_{-\infty}^{\infty} \varphi(t)\mathrm{e}^{-j\omega t}\mathrm{d}t \tag{1}$$

where $\omega$ is a constant, and $F$ represents the Fourier integral transform operator.

The inverse Fourier integral transform is given as [18]:

$$\varphi(t) = \mathbb{Z}^{-1}\left[\varphi(\eta)\right] = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \varphi(\eta)\eta\mathrm{e}^{j\eta t}\mathrm{d}\eta \tag{2}$$

where $F^{-1}$ represents the inverse Fourier integral transform operator.

The Fourier-Yang integral transform of the function $\varphi(t)$ is shown as [19]:

$$\Phi(\eta)=\mathbb{Z}\left[\varphi(t)\right]=\frac{1}{\eta}\int_{-\infty}^{\infty} \Sigma(t)\mathrm{e}^{-j\eta t}\mathrm{d}t \tag{3}$$

where $\eta$ is a real-valued constant, $\mathbb{Z}$ represents the Fourier-Yang integral transform operator.

The inverse Fourier-Yang integral transform is given by [19]:

$$\varphi(t) = \mathbb{Z}^{-1}\left[\varphi(\eta)\right] = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \varphi(\eta)\eta\mathrm{e}^{j\eta t}\mathrm{d}\eta \tag{4}$$

where $\mathbb{Z}^{-1}$ represents the inverse Fourier-Yang integral transform operator.

Substituting eq. (3) into eq. (4), we have the integral criterion as:

$$\varphi(t) = \mathbb{Z}^{-1}\left[\Phi(\eta)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta\left[\frac{1}{\eta}\int_{-\infty}^{\infty} \varphi(t)\mathrm{e}^{j\eta t}\mathrm{d}t\right]\mathrm{e}^{j\eta t}\mathrm{d}\eta \tag{5}$$

The properties of the Fourier-Yang integral transform are as:

(T1) $\rightarrow$ $\varphi(\omega) = F[\varphi(t)]$ and $\Phi(\eta) = \mathbb{Z}[\varphi(t)]$, then we have:

$$\varphi(\omega) = \eta\Phi(\eta) \tag{6}$$

$$\Phi(\eta) = \frac{1}{\omega}\varphi(\omega) \tag{7}$$

(T2) $\rightarrow$ $\Phi(\eta) = \mathbb{Z}[\varphi(t)]$ and $\Psi(\eta) = \mathbb{Z}[\Psi(t)]$ then we have:

$$\mathbb{Z}\left[a\varphi(t) + b\Psi(t)\right] = a\Phi(\eta) + b\Psi(\eta) \tag{8}$$

where $a$ and $b$ are the constants.

(T3) $\rightarrow$ $\Phi(\eta) = \mathbb{Z}[\varphi(t)]$, then we have:

$$\mathbb{Z}\left[\varphi(t-a)\right] = \mathrm{e}^{j\eta a}\Phi(\eta) \tag{9}$$

where $a$ is a constant.

(T4) $\rightarrow$ $\Phi(\eta) = \mathbb{Z}[\varphi(t)]$, then we have:

$$\mathbb{Z}\left[\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right] = j\eta\Phi(\eta) \tag{10}$$

(T5) $\rightarrow$ $\Phi(\eta) = \mathbb{Z}[\varphi(t)]$, then we have:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty} \varphi(t)\mathrm{d}t\right]=\frac{1}{j\eta}\Phi(\eta) \tag{11}$$

(T6) → $\Phi'(\eta)$ =d$\Phi(\eta)$/d($\eta$) hen we have:

$$\Phi'(\eta)=-\frac{1}{\eta}\Phi(\eta)-j\eta\Phi(\eta) \tag{12}$$

(T7) → $\Phi(\eta)=\mathbb{Z}[\varphi(t)]$ and $\Psi(\eta)=\mathbb{Z}[\Psi(t)]$, then we have the Fourier-Yang integral transform of convolution:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\Psi(\tau)\mathrm{d}t\right]=\eta\Psi(\eta)\Phi(\eta) \tag{13}$$

(T8) Let $\varphi(t)=\mathrm{e}^{-at}v(t)$, where $v(t)$ is the Heaviside unit step function [18]. Then:

$$\Phi(\eta)=\frac{1}{\eta(a+j\eta)} \tag{14}$$

where $a$ is a constant.
(T9) Let $\varphi(t)=\delta(t)$, where $\delta(t)$ is the Dirac function. Then:

$$\Phi(\eta)=\frac{1}{\eta} \tag{15}$$

(T10) Let $\varphi(t)=\begin{cases}C & |t|<T\\ 0 & \text{else}\end{cases}$ where $C$ and $T$ are constants. Then:

$$\mathbb{Z}[\varphi(t)]=\frac{2\sin(\eta T)}{\eta^2} \tag{16}$$

(T11) Let $\varphi(t)=\upsilon\mathrm{e}^{-kt^2}, k>0,$ where $\upsilon$ is a constant. Then:

$$\mathbb{Z}[\varphi(t)]=\frac{\upsilon\pi}{\eta\sqrt{k\pi}}\mathrm{e}^{-\frac{\eta^2}{4k}} \tag{17}$$

*Proof*.
(T1) Taking $\omega=\eta$ in eqs. (1) and (3), we obtain:

$$\varphi(\omega)=F[\varphi(t)]=\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\omega t}\mathrm{d}t=\eta\left[\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t\right]=\eta\Phi(\eta) \tag{18}$$

$$\Phi(\eta)=\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=\frac{1}{\omega}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\omega t}\mathrm{d}t=\varphi(\omega) \tag{19}$$

(T2)

$$\mathbb{Z}[a\varphi(t)+b\Psi(t)]=\frac{1}{\eta}\int_{-\infty}^{\infty}a\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t+\frac{1}{\eta}\int_{-\infty}^{\infty}b\Psi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=a\Phi(\eta)+b\Psi(\eta) \tag{20}$$

(T3)

$$\mathbb{Z}[\varphi(t-a)]=\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t-a)\mathrm{e}^{-j\eta t}\mathrm{d}t \tag{21}$$

If $t=t-a$, then:

$$\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{e}^{j\eta a}\mathrm{d}t=\mathrm{e}^{j\eta a}\frac{1}{\eta}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=\mathrm{e}^{j\eta a}\Phi(\eta) \tag{22}$$

(T4)

$$\mathbb{Z}\left[\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}\mathrm{e}^{-j\eta t}\mathrm{d}t=\frac{1}{\eta}\int_{-\infty}^{\infty}\mathrm{e}^{-j\eta t}\mathrm{d}\varphi(t)=j\eta\Phi(\eta) \tag{23}$$

Similarly, we have:

$$\mathbb{Z}\left[\frac{\mathrm{d}^n\varphi(t)}{\mathrm{d}t^n}\right]=j^n\eta^n\Phi(\eta) \tag{24}$$

where $n$ is the positive integer.

(T5)

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t)\,\mathrm{d}t\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(t)\,\mathrm{d}t\right]\mathrm{e}^{-j\eta t}\mathrm{d}t=\frac{1}{j\eta}\Phi(\eta) \tag{25}$$

Similarly, we have:

$$\mathbb{Z}\left[\int_1\cdots\int_n\varphi(t)\,\mathrm{d}t\right]=\frac{1}{j^n\eta^n}\Phi(\eta) \tag{26}$$

(T6)

$$\Phi'(\eta)=-\frac{1}{\eta^2}\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t-j\int_{-\infty}^{\infty}\varphi(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=-\frac{1}{\eta}\Phi(\eta)-j\eta\Phi(\eta) \tag{27}$$

(T7)

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\Psi(\tau)\,\mathrm{d}t\right]=\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)\left[\int_{-\infty}^{\infty}\varphi(t-\tau)\mathrm{e}^{-j\eta t}\mathrm{d}t\right]\mathrm{d}\tau \tag{28}$$

If $\lambda = t-\tau$, then:

$$\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)\left[\int_{-\infty}^{\infty}\varphi(\lambda)\mathrm{e}^{-j(\lambda+\tau)\eta}\mathrm{d}\lambda\right]\mathrm{d}\tau=\frac{1}{\eta}\int_{-\infty}^{\infty}\Psi(\tau)\mathrm{e}^{-j\tau\eta}\mathrm{d}\tau\int_{-\infty}^{\infty}\varphi(\lambda)\mathrm{e}^{-j\lambda\eta}\mathrm{d}\lambda=\eta\Psi(\eta)\varphi(\eta) \tag{29}$$

Similarly, we have:

$$\mathbb{Z}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t_1-\tau_1,t_2-\tau_2)\Psi(\tau_1,\tau_2)\mathrm{d}t_1\mathrm{d}t_2\right]=\eta_1\eta_2\Psi(\eta_1,\eta_2)\varphi(\eta_1,\eta_2) \tag{30}$$

(T8) If the Heaviside unit step function [18] is:

$$\int_{-\infty}^{\infty}\phi(t)\nu(t)\mathrm{d}t=\int_0^{\infty}\phi(t)\,\mathrm{d}t \tag{31}$$

then:

$$\Phi(\eta)=\mathbb{Z}[\varphi(t)]=\frac{1}{\eta}\int_{-\infty}^{\infty}\mathrm{e}^{-at}\nu(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=\frac{1}{\eta}\int_0^{\infty}\mathrm{e}^{-(a+j\eta)t}\mathrm{d}t=\frac{-1}{\eta(a+j\eta)}\mathrm{e}^{-(a+j\eta)t}\Big|_0^{\infty}=\frac{1}{\eta(a+j\eta)} \tag{32}$$

(T9) If the Dirac function is:

$$\delta(\tau-c)=\begin{cases}\infty, & \tau=c\\ 0, & \tau\neq c\end{cases},\quad \int_{-\infty}^{+\infty}\delta(\tau-c)f(\tau)\,\mathrm{d}\tau=f(c) \tag{33}$$

then:

$$\Phi(\eta)=\mathbb{Z}[\varphi(t)]=\frac{1}{\eta}\int_{-\infty}^{\infty}\delta(t)\mathrm{e}^{-j\eta t}\mathrm{d}t=\frac{1}{\eta} \tag{34}$$

(T10)

$$\mathbb{Z}[\varphi(t)]=\frac{2}{\eta}\int_0^{\mathrm{T}}C\cos(\eta t)\mathrm{d}t=\frac{2\sin(\eta T)}{\eta^2} \tag{35}$$

(T11)

$$\Sigma(\eta)=\frac{1}{\eta}\int_{-\infty}^{\infty}\upsilon e^{-kt^2}e^{-j\eta t}dt=\frac{\upsilon}{\eta}\int_{-\infty}^{\infty}e^{\left[-k\left(t+\frac{j\eta}{2k}\right)^2-\frac{\eta^2}{4k}\right]}dt \tag{36}$$

If $t = t + j\eta/2k$, then we get:

$$\mathbb{Z}\left[\upsilon e^{-kt^2}\right]=\frac{\upsilon}{\eta}e^{-\frac{\eta^2}{4k}}\int_{-\infty}^{\infty}e^{-kt^2}dt=\frac{\upsilon\pi}{\eta\sqrt{k\pi}}e^{-\frac{\eta^2}{4k}} \tag{37}$$

where:

$$\int_{-\infty}^{\infty}e^{-kt^2}dt=\sqrt{\frac{\pi}{k}} \tag{38}$$

## Applications

### *Solving the 1-D wave equation*

In this section, with the help of the Fourier-Yang integral transform, the analytical solution of the 1-D wave equation is shown:

The mathematical model of 1-D wave equation is defined [20]:

$$\frac{\partial^2\varphi(x,t)}{\partial t^2}-\lambda^2\frac{\partial^2\varphi(x,t)}{\partial x^2}=0,\quad(-\infty<x<\infty,t>0) \tag{39}$$

where $\lambda$ is a constant.

The initial conditions are given by:

$$\varphi(x,0)=\vartheta(x),\quad\frac{\partial\varphi(x,0)}{\partial t}=0 \tag{40}$$

Using the eqs. (3) and (24), the Fourier-Yang integral transforms of eq. (39) with respect to $x$ are given:

$$\frac{\partial^2\Phi(\eta,t)}{\partial t^2}+\lambda^2\eta^2\Phi(\eta,t)=0 \tag{41}$$

Similarly, the initial conditions becomes:

$$\Phi(\eta,0)=\vartheta(\eta),\quad\frac{\partial\Phi(\eta,0)}{\partial t}=0 \tag{42}$$

Making use of eqs. (41) and (42), we have:

$$\Sigma(\eta,t)=\vartheta(\eta)\cos(\lambda\eta t) \tag{43}$$

Substituting eq. (43) into eq. (4), and with the help of eq. (9), we obtain:

$$\varphi(x,t)=\mathbb{Z}^{-1}\left[\Phi(\eta,t)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta\vartheta(\eta)\cos(\lambda\eta t)e^{j\eta t}d\eta=\frac{1}{2}\left[\vartheta(x-\lambda t)+\vartheta(x+\lambda t)\right] \tag{44}$$

the solution of eq. (44) is identical to the result [20].

### *Solving the 2-D heat transfer equation*

The PDE in the 2-D heat transfer problem is:

$$\frac{\partial\varphi(x,y,t)}{\partial t}-\kappa^2\left[\frac{\partial^2\varphi(x,y,t)}{\partial x^2}+\frac{\partial^2\varphi(x,y,t)}{\partial y^2}\right]=0,\ (-\infty<x,y<\infty,t>0) \tag{45}$$

where $\kappa$ is the thermal conductivity.

The initial condition is:

$$\varphi(x,y,0)=\vartheta(x,y) \tag{46}$$

Combining eqs. (3) and (24) yields the Fourier-Yang integral transforms of eq. (45) with respect to $x$ and $y$ expressed:

$$\frac{\partial \Phi(\eta_1,\eta_2,t)}{\partial t}+\kappa^2\left(\eta_1^{\;2}+\eta_2^{\;2}\right)\Phi(\eta_1,\eta_2,t)=0 \tag{47}$$

Similarly, the initial conditions is：

$$\Phi(\eta_1,\eta_2,0)=\vartheta(\eta_1,\eta_2) \tag{48}$$

From eqs. (47) and (48), we have:

$$\Phi(\eta_1,\eta_2,t)=\eta_1\eta_2\vartheta(\eta_1,\eta_2)\Psi(\eta_1,\eta_2) \tag{49}$$

where

$$\Psi(\eta_1,\eta_2)=\frac{1}{\eta_1}\frac{1}{\eta_2}\mathrm{e}^{-\kappa^2\left(\eta_1^{\;2}+\eta_2^{\;2}\right)t} \tag{50}$$

Utilizing eqs. (4) and (37), the inverse Fourier-Yang integral transform of eq. (50) is written:

$$\mathbb{Z}^{-1}\left[\frac{1}{\eta_1}\frac{1}{\eta_2}\mathrm{e}^{-\kappa^2\left(\eta_1^{\;2}+\eta_2^{\;2}\right)t}\right]=\left[\frac{1}{2\pi\kappa}\sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{x^2}{4\kappa^2 t}}\right]\left[\frac{1}{2\pi\kappa}\sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{y^2}{4\kappa^2 t}}\right]=\frac{1}{4\pi\kappa^2 t}\mathrm{e}^{\frac{-\left(x^2+y^2\right)}{4\kappa^2 t}} \tag{51}$$

Substution of eqs. (50) and (51) into eq. (30), we have the analytical solution of the 2-D heat transfer problem as:

$$\Phi(x,y,t)=\frac{1}{4\pi\kappa^2 t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vartheta(\tau_1,\tau_2)\mathrm{e}^{\frac{-\left[(x-\tau_1)^2+(y-\tau_2)^2\right]}{4\kappa^2 t}}\mathrm{d}\tau_1\mathrm{d}\tau_2 \tag{52}$$

## Conclusion

In this work, some new properties of the Fourier-Yang integral transform are extended firstly, and the integral transforms of some functions are given. Applying those, we obtain the analytical solutions of the differential equations in the 1-D wave and the 2-D heat transfer problems. The results indicate that the Fourier-Yang integral transform is effective and precise in solving the partial differential equations.

## Nomenclature

$t$ – time, [s]
$x, y$ – space co-ordinate, [m]

*Greek symbols*

$\lambda$ – wave propagation rate, [$ms^{-1}$]
$\kappa$ – thermal conductivity, [$Wm^{-2}K^{-1}$]

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