

ANALYTICAL SOLUTIONS OF BIHARMONIC EQUATION BY THE FOURIER-YANG INTEGRAL TRANSFORM

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The biharmonic equation are frequently encountered in CFD. In this investigation, the biharmonic equation in the semi-infinite domains is addressed using a new Fourier-like integral transform proposed in [1]. The properties of the new Fourier-like integral transform are expanded in this article. Meanwhile, the analytical solutions for the biharmonic equation in the semi-infinite domains are found. This demonstrates the new Fourier-like integral transform is an efficient and accurate method to clarify mathematical physics problems described by PDE.

Key words: *biharmonic equation, Laplace equation, PDE, Fourier-Yang integral transform, fluid dynamics*

Introduction

The biharmonic equation (BE) derived from the mathematical model in fluid dynamics have an extensive application background in science research and engineering technology. Accordingly, there is growing concern about the improvement on calculation method for BE [2-7]. Mathieu [8] suggested the method of successive approximations for solving the BE and proved its convergence for a square plate. Using the Jacobi-Galerkin approximations, Doha and Bhrawy [9] solved the BE under the first and second boundary conditions. The 2-D BE for the semi-infinite elastic medium in the absence of the body force was settled [10]. Those methods effectively solved the BE for certain boundary conditions. However, some boundary conditions are still challenging to deal with, thus the method of solving BE under certain boundary conditions is also need to continuously improve and innovate.

The integral transform (IT) has become an increasingly widespread application in solving the PDE. Currently, the conventional IT of three types, *e. g.*, Laplace transform (LT) [11, 12], Fourier transform [13, 14], and Sumudu transform (ST) [11, 15], were widely used in solving the PDE arising from mathematical physics. In recent years, many scholars also applied IT to solve the BE [10, 16, 17]. Still, there are unresolved boundary question for actual engineering. In view of the aforementioned facts, we have focused on a new Fourier-like IT termed Fourier-Yang IT proposed [1] to handle PDE on steady heat transfer. The proposed IT was subsequently applied to solved the 1-D heat diffusion equation [18]. Those applications demonstrate that the Fourier-Yang integral transform can serve as a new effective method for solution

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of the PDE. Therefore, the object of this paper is to solve the BE in semi-infinite medium based on the Fourier-Yang integral transform.

The new Fourier-like integral transforms

The Fourier-like integral transform of $\theta(t)$, denoted by $\tilde{\theta}(\kappa)$, is defined as, see [1]:

$$\Gamma[\theta(t)] = \kappa \int_{-\infty}^{+\infty} \theta(t) e^{-i\kappa t} dt \quad (1)$$

where Γ is called the Fourier-like integral transform operator or the Fourier-Yang transform operator.

The inverse Fourier-like integral transform of $\tilde{\theta}(\kappa)$ is defined by:

$$\Gamma^{-1}[\tilde{\theta}_2(\kappa)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{\theta}_2(\kappa)}{\kappa} e^{i\kappa t} d\kappa \quad (2)$$

The properties of the Fourier-like integral transform are presented;

(T1) If $\tilde{\theta}_1(\kappa) = \Gamma[\theta_1(t)]$ and $\tilde{\theta}_2(\kappa) = \Gamma[\theta_2(t)]$ then we have [1]:

$$\Gamma[a\theta_1(t) + b\theta_2(t)] = a\tilde{\theta}_1(\kappa) + b\tilde{\theta}_2(\kappa) \quad (3)$$

where a and b are constants.

(T2) Let $\Gamma[\theta(t)] = \tilde{\theta}(\kappa)$. Then we have:

$$\Gamma[\theta(at)] = \tilde{\theta}\left(\frac{\kappa}{a}\right) \quad (4)$$

where a is a constant greater than zero.

Proof. We present:

$$\Gamma[\theta(at)] = \kappa \int_{-\infty}^{+\infty} \theta(at) e^{-i\kappa t} dt \quad (5)$$

Taking $x = at$, we have:

$$\Gamma[\theta(at)] = \frac{\kappa}{a} \int_{-\infty}^{+\infty} \theta(x) e^{-i\frac{\kappa}{a}x} dx = \tilde{\theta}\left(\frac{\kappa}{a}\right) \quad (6)$$

For $a < 0$, it is easily obtained that:

$$\Gamma[\theta(at)] = -\tilde{\theta}\left(\frac{\kappa}{a}\right) \quad (7)$$

(T3) Let $\Gamma[\theta(t)] = \tilde{\theta}(\kappa)$. Then we have:

$$\Gamma[\theta(t \pm t_0)] = e^{\pm i\kappa t_0} \tilde{\theta}(\kappa) \quad (8)$$

Proof. We have:

$$\Gamma[\theta(t \pm t_0)] = \kappa \int_{-\infty}^{+\infty} \theta(t \pm t_0) e^{-i\kappa t} dt \quad (9)$$

Taking $x = at$ gives:

$$\begin{aligned} \Gamma[\theta(t \pm t_0)] &= \kappa \int_{-\infty}^{+\infty} \theta(x) e^{-i\kappa(x \pm t_0)} dx \\ &= e^{\pm i\kappa t_0} \kappa \int_{-\infty}^{+\infty} \theta(x) e^{-i\kappa x} dx \\ &= e^{\pm i\kappa t_0} \tilde{\theta}(\kappa) \end{aligned} \quad (10)$$

(T4) Let $\Gamma[\mathcal{G}(t)] = \tilde{\mathcal{G}}(\kappa)$ and $\lim_{t \rightarrow -\infty} f^{(m)}(t) = 0 (m = 0, 1, \dots, n-1)$. Then:

$$\Gamma[\mathcal{G}^{(n)}(t)] = (i\kappa)^n \tilde{\mathcal{G}}(\kappa) \quad (11)$$

where $\mathcal{G}^{(n)}(t)$ is the derivative of \mathcal{G} of order n with respect to t .

Proof. For $n = 1$, we have [18]:

$$\begin{aligned} \Gamma[\mathcal{G}^{(1)}(t)] &= \kappa \int_{-\infty}^{+\infty} \mathcal{G}^{(1)}(t) e^{-i\kappa t} dt \\ &= \kappa \mathcal{G}(t) e^{-i\kappa t} \Big|_{-\infty}^{+\infty} + i\kappa^2 \int_{-\infty}^{+\infty} \mathcal{G}(t) e^{-i\kappa t} dt \\ &= i\kappa \tilde{\mathcal{G}}(\kappa) \end{aligned} \quad (12)$$

Generally:

$$\Gamma[\mathcal{G}^{(n)}(t)] = i\kappa \Gamma[\mathcal{G}^{(n-1)}(t)] = (i\kappa)^2 \Gamma[\mathcal{G}^{(n-2)}(t)] = (i\kappa)^n \tilde{\mathcal{G}}(\kappa) \quad (13)$$

(T5) Suppose $\Gamma[\mathcal{G}(t)] = \tilde{\mathcal{G}}(\kappa)$ and $\theta(t) = \int_{-\infty}^t \mathcal{G}(t) dt$. Then we have:

$$\Gamma[\theta(t)] = \frac{1}{i\kappa} \tilde{\mathcal{G}}(\kappa) \quad (14)$$

Proof. We get:

$$\begin{aligned} \Gamma[\theta(t)] &= \kappa \int_{-\infty}^{+\infty} \theta(t) e^{-i\kappa t} dt \\ &= -\frac{1}{i} \theta(t) e^{-i\kappa t} \Big|_{-\infty}^{+\infty} + \frac{1}{i} \int_{-\infty}^{+\infty} \mathcal{G}(t) e^{-i\kappa t} dt \\ &= \frac{1}{i\kappa} \tilde{\mathcal{G}}(\kappa) \end{aligned} \quad (15)$$

(T6) If $\tilde{\mathcal{G}}_1(\kappa) = \Gamma[\mathcal{G}_1(t)]$ and $\tilde{\mathcal{G}}_2(\kappa) = \Gamma[\mathcal{G}_2(t)]$, then we have:

$$\Gamma[\mathcal{G}_1(t) \mathcal{G}_2(t)] = \frac{1}{\kappa} \tilde{\mathcal{G}}_1(\kappa) \tilde{\mathcal{G}}_2(\kappa) \quad (16)$$

where the convolution of $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$ is defined:

$$\mathcal{G}_1(t) \mathcal{G}_2(t) = \int_{-\infty}^{+\infty} \mathcal{G}_1(x) \mathcal{G}_2(t-x) dx \quad (17)$$

Proof. We have:

$$\begin{aligned}
\Gamma[\mathcal{G}_1(t)\mathcal{G}_2(t)] &= \kappa \int_{-\infty}^{+\infty} (\mathcal{G}_1\mathcal{G}_2)(t)e^{-ikt} dt \\
&= \kappa \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \mathcal{G}_1(x)\mathcal{G}_2(t-x) dx \right] e^{-ikt} dt \\
&= \kappa \int_{-\infty}^{+\infty} \mathcal{G}_1(x) \left[\int_{-\infty}^{+\infty} \mathcal{G}_2(t-x)e^{-ikt} dt \right] dx \\
&= \int_{-\infty}^{+\infty} \mathcal{G}_1(x)\tilde{\mathcal{G}}_2(\kappa)e^{-ikx} dx \\
&= \frac{1}{\kappa} \tilde{\mathcal{G}}_1(\kappa)\tilde{\mathcal{G}}_2(\kappa)
\end{aligned} \tag{18}$$

(T7) Let $\mathcal{G}(t) = \delta(t)$. Then, see [1]:

$$\tilde{\mathcal{G}}_1(\kappa) = \Gamma[\delta(t)] = k \tag{19}$$

where $\delta(t)$ is the Dirac function.

(T8) If $\mathcal{G}(t) = \text{sgn}(t)$, then we have:

$$\tilde{\mathcal{G}}_1(\kappa) = \Gamma[\text{sgn}(t)] = -2i \tag{20}$$

where $\text{sgn}(t)$ respects the sign function defined by [10]:

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{21}$$

Proof. With the aid of exponential decay function defined by [19]:

$$f(t) = e^{at}u(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}, \quad a > 0 \tag{22}$$

it is easily obtained:

$$\Gamma[f(t)] = k \int_0^{+\infty} e^{-(a+ik)t} dt = \frac{k}{a+ik} \tag{23}$$

Meanwhile, the sign function $\text{sgn}(t)$ can be written as, see [20]:

$$\text{sgn}(t) = \lim_{a \rightarrow 0} e^{-at}u(t) - \lim_{a \rightarrow 0} e^{-at}u(-t) \tag{24}$$

Hence:

$$\Gamma[\text{sgn}(t)] = \lim_{a \rightarrow 0} \frac{k}{a+ik} - \lim_{a \rightarrow 0} \frac{k}{a-ik} = \frac{2}{i} \tag{25}$$

Solving PDE in semi-infinite domains

We consider the BE in the semi-infinite domain, see [10]:

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0, \quad y > 0 \quad (26)$$

which can be presented symbolically:

$$\nabla^4 \psi = 0 \quad (27)$$

subject to the boundary conditions:

$$\frac{\partial^2 \psi}{\partial y^2} = -p(y), \quad \frac{\partial^2 \psi}{\partial xy} = 0, \quad x = 0 \quad (28)$$

By taking the Fourier-Yang IT with respect to y , we present in the form:

$$\left(\frac{d^2}{dx^2} - \kappa^2 \right)^2 \psi = 0 \quad (29)$$

subject to the initial value:

$$\kappa^2 \tilde{\psi}(0, \kappa) = \tilde{p}(\kappa), \quad (i\kappa) \left(\frac{d\tilde{\psi}}{dx} \right)_{x=0} = 0 \quad (30)$$

where $\Gamma[\psi(x, y)] = \tilde{\psi}(x, \kappa)$ and $\Gamma[p(y)] = \tilde{p}(\kappa)$.

In view of eq. (29), we obtain its bounded solution in the following form:

$$\tilde{\psi}(x, \kappa) = (A + Bx) \exp(-|\kappa|x) \quad (31)$$

where A and B are constants to be determined.

It follows from eq. (30):

$$A = \frac{\tilde{p}(\kappa)}{\kappa^2}, \quad B = \frac{\tilde{p}(\kappa)}{|\kappa|} \quad (32)$$

Hence, eq. (31) can be expressed:

$$\tilde{\psi}(x, \kappa) = \frac{\tilde{p}(\kappa)}{\kappa^2} (1 + |\kappa|x) \exp(-|\kappa|x) \quad (33)$$

Application of the inverse transform of the new IT for eq. (33) gives:

$$\psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\tilde{p}(k)}{\kappa^3} (1 + |\kappa|x) \exp(i\kappa y - |\kappa|x) d\kappa \quad (34)$$

In particular:

$$p(y) = P_0 \delta(y) \quad (35)$$

it is easily obtained from eq. (19):

$$\tilde{p}(k) = P_0 k \quad (36)$$

where P_0 is a constant.

Substituting this value into eq. (34), we obtain:

$$\begin{aligned}\psi(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{P_0}{\kappa^2} (1 + |\kappa|x) \exp(i\kappa y - |\kappa|x) d\kappa \\ &= \frac{P_0}{\pi} \int_0^{+\infty} \frac{1}{\kappa^2} (1 + \kappa x) \cos(\kappa y) \exp(-\kappa x) d\kappa\end{aligned}\quad (37)$$

This result is agreement with the solution in [10].

Another special case is:

$$p(y) = \operatorname{sgn}(y) \quad (38)$$

so that:

$$\tilde{p}(k) = -2i \quad (39)$$

In view of eq. (34), the solution becomes:

$$\begin{aligned}\psi(x, y) &= \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\kappa^2} (1 + |\kappa|x) \exp(i\kappa y - |\kappa|x) d\kappa \\ &= \frac{2i}{\pi} \int_0^{+\infty} \frac{1}{\kappa^2} (1 + \kappa x) \cos(\kappa y) \exp(-\kappa x) d\kappa\end{aligned}\quad (40)$$

Conclusion

In this paper, some properties of the new IT were developed. It was used to find the analytical solutions of the BE in semi-infinite domains. The results are in agreement with those of Fourier transform and Laplace transform. The technology can solve the PDE in mathematical physics accurately and efficiently.

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Nomenclature

a – the space co-ordinate, [m]
 $\operatorname{sgn}(t)$ – the signal function
 t – time, [s]
 y – the space co-ordinate, [m]

x – the space co-ordinate, [m]

Greek symbol

$\delta(t)$ – the Dirac function

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