THE SOLUTION OF LOCAL FRACTIONAL DIFFUSION EQUATION INVOLVING HILFER FRACTIONAL DERIVATIVE

by

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In this present work the Yang-Fourier transform method incorporating the Laplace transform method is used to solve fractional diffusion equations involving the Hilfer fractional derivative and local fractional operator. The exact solution is obtained.

Key words: *diffusion equations, Yang-Fourier transform, Laplace transform, Hilfer fractional derivative, local fractional operator*

Introduction

Nowadays, fractional calculus receives increasing attention in the scientific community. Fractional differential equation have become a useful tool in many areas of sciences and technologies. For example, fractional diffusion equation has been successfully used to describe important physical and chemical phenomena [1-6].

In recent years, the fractional diffusion equations involving the Hilfer fractional derivative and local fractional operator appear frequently in the literature [7-11]. The Hilfer fractional derivative was first proposed by Hilfer in the analysis of fractional diffusion equations, obtaining that such kind of models can be used in context of glass relaxation and aquifer problems. The theory of local fractional derivative have applied in Fokker-Planck equations, Helmholtz equations and the fractal dynamical systems, *etc*. The main purpose of using the local fractional order derivative is to describe the anomalous diffusion phenomena in fractal media [12-15].

In this paper, we consider the following fractional order reaction-diffusion equation:

$$\mathrm{D}_t^{\beta,\gamma}u(x,t)=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u(x,t) \tag{1}$$

subject to the initial condition:

$$J^{(1-\beta)(1-\gamma)}u(x,0)=f(x) \tag{2}$$

where $\mathrm{D}^{\beta,\gamma}$ is a Hilfer fractional derivative of order $0<\beta\le 1$ and type $0\le\gamma\le 1$, J^{λ} – the Riemann-Liourille fractional integral of order λ, $\partial^{2\alpha}/\partial x^{2\alpha}$ denotes the local fractional derivative of order 2α and $f(x)$ is a given function of x.

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Fractional calculus and local fractional calculus

In this section, we recall some basic definitions and properties on the Hilfer fractional derivative and the local fractional derivative which shall be used in this paper. For more detail see [8, 16] .

The Riemann-Liouville (R-L) fractional integral operator of orde $\alpha > 0$ of a function is defined:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) \mathrm{d}s \tag{3}$$

$$J^0 f(x) = f(x) \tag{4}$$

Properties of the operator J^{α} can be found in [16] and we mention only the following. For $\alpha, \beta \geq 0$, $x > 0$, and $\lambda > -1$:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x) \tag{5}$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x) \tag{6}$$

$$J^{\alpha}(x^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(1+\alpha+\lambda)} x^{\lambda+\alpha} \tag{7}$$

The fractional derivative of $f(x)$ in Riemann-Liouville sense is defined by:

$${}^{RL}\mathrm{D}^{\mu} f(y) = \frac{d^n}{\mathrm{d}y^n} J^{n-\mu} f(y), \quad n = [\mathrm{Re}(\mu)] + 1 \tag{8}$$

The Caputo fractional derivative is defined by:

$${}^{C}\mathrm{D}^{\mu} f(y) = J^{n-\mu} f^{(n)}(y) \tag{9}$$

We recall the following basic properties:

$${}^{C}\mathrm{D}^{\alpha} J^{\alpha} f(x) = f(x) \tag{10}$$

$$J^{\alpha}\, {}^{C}\mathrm{D}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0 \tag{11}$$

Hilfer generalized the fractional derivative (8) by the following fractional derivative of order $0 < \beta \leq 1$ and type $0 \leq \gamma \leq 1$:

$$\mathrm{D}^{\beta,\gamma} f(y) = J^{\gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}y} J^{(1-\gamma)(1-\beta)} f(y) \tag{12}$$

which is called Hilfer fractional derivative. Nate that when $0 < \beta \leq 1$, $\gamma = 0$, the Hilfer fractional derivative would to the R-L fractional derivative:

$${}^{\mathrm{RL}}\mathrm{D}^{\beta} f(y) = \frac{\mathrm{d}}{\mathrm{d}y} J^{1-\beta} f(y) \tag{13}$$

when $0 < \beta \leq 1$, $\gamma = 1$, it corresponds to the Caputo fractional derivative:

$${}^{C}\mathrm{D}^{\beta} f(y) = J^{1-\beta} f'(y) \tag{14}$$

On the Laplace transform of the Hilfer fractional derivative (12), in [8], it is found for $0 < \beta < 1$:

$$L\left[\mathrm{D}^{\beta,\gamma} f(y)\right] = s^{\beta} L\left[f(y)\right] - s^{\gamma(\beta-1)} J^{(1-\beta)(1-\gamma)} f(0)$$

where the initial value term:

$$J^{(1-\beta)(1-\gamma)} f(0) \tag{15}$$

is evaluated in the limit $y \to 0^+$, in the space of the Lebesgue integrable functions:

$$L[0,\infty] = \left\{ f : \|f\|_1 = \int_0^{\infty} |f(y)| \mathrm{d}y < \infty \right\}$$

Next, we recall the notations and some properties of the local fractional operators [16, 17].

The function $f(x)$ is called the local fractional continuous (LFC) at $x = x_0$, if it is valid:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha}, \quad 0 < \alpha \le 1 \tag{16}$$

with $|x - x_0| < \delta$, for $\varepsilon > 0$. If $f(x)$ is the LFC for any $x \in (a, b)$ then it is called LFC on the interval (a, b) denoted by $f(x) \in C_{\alpha}(a, b)$.

Setting $f(x) \in C_{\alpha}(a, b)$ local fractional derivative (LFD) of $f(x)$ at $x = x_0$ is defined:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}} f(x)|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x - x_0)^{\alpha}} \tag{17}$$

where $\Delta^{\alpha}[f(x) - f(x_0)] \cong \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$

The local fractional integral (LFI) of $f(x)$ in the interval $[a, b]$ is defined:

$$\int_a^b f(t)(\mathrm{d}t)^{\alpha} = \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha} \tag{18}$$

where $\{t_j \mid j = 0, 1, \ldots N\}$, is a partition of the interval $[a, b]$, $\Delta t_j = t_j - t_{j-1}$, $\Delta t = \{\Delta t_j\}$, $t_0 = a$, and $t_N = b$.

If $f(x) \in C_{\alpha}(a, b)$ we define its Yang-Fourier transform:

$$F_{\alpha}[f(x);\omega] = f_{\omega}^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{+\infty} E_{\alpha}(-i^{\alpha}\omega^{\alpha}x^{\alpha}) f(x)(\mathrm{d}x)^{\alpha} \tag{19}$$

and its inverse Yang-Fourier transform:

$$f(x) = F_{\alpha}^{-1}[f_{\alpha}^{F,\alpha}(\omega)] = \frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{+\infty} E_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha}) f_{\alpha}^{F,\alpha}(\omega)(\mathrm{d}\omega)^{\alpha} \tag{20}$$

Here, $E_{\alpha}(z)$ is the Mittag-Leffer function:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{21}$$

We also use the following two-parameter Mittag-Leffer function in the next section:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{22}$$

The following formulas are valid:

$$F_{\alpha}[af(x) + bg(x);\omega] = aF_{\alpha}[f(x);\omega] + bF_{\alpha}[g(x);\omega] \tag{23}$$

$$F_\alpha\left[\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}f(x);\omega\right]=i^\alpha\omega^\alpha F[f(x);\omega] \tag{24}$$

$$L\left[t^{(\mu-1)(1-\nu)}E_{\mu,1-(1-\mu)(1-\nu)}(-\lambda t^\mu)\right]=\frac{s^{-\nu(1-\mu)}}{s^\mu+\lambda} \tag{25}$$

The solution of the problem (1)-(2)

In this section, we investigate the problem (1)-(2). Our main result:

The solution of the problem (1)-(2):

$$u(x,t)=\int_{-\infty}^{+\infty}f(\xi)v(x-\xi,t)(\mathrm{d}\xi)^\alpha \tag{26}$$

where

$$v(x,t)=F_\alpha^{-1}[g(\omega,t);x] \tag{27}$$

and

$$g(\omega,t)=t^{(\beta-1)(1-\gamma)}E_{\beta,\beta-\gamma(\beta-1)}(-t^\beta i^\alpha\omega^\alpha) \tag{28}$$

Let $u^*(\omega, t)$ denotes the Yang-Fourier transform of the function $u(x, t)$ with respect to x. Taking the Yang-Fourier transform of both sides of eq. (1) yields:

$$\mathrm{D}_t^{\beta,\gamma}u^*(\omega,t)=i^\alpha\omega^\alpha u^*(\omega,t) \tag{29}$$

Applying the Laplace transform to (27), we obtain:

$$s^\beta L[u^*(\omega,t);s]-s^{\gamma(\beta-1)}F_\alpha[f(x);\omega]=i^\alpha\omega^\alpha L[u^*(\omega,t);s]$$

So:

$$L[u^*(\omega,t);s]=\frac{s^{\gamma(\beta-1)}f^*(\omega)}{s^\beta-i^\alpha\omega^\alpha} \tag{30}$$

where

$$f^*(\omega)=F_\alpha[f(x);\omega]$$

From the inverse Laplace transform of the relation (30), and the (27), we get:

$$u^*(\omega,t)=f^*(\omega)t^{(\beta-1)(1-\gamma)}E_{\beta,\beta-\gamma(\beta-1)}(t^\beta i^\alpha\omega^\alpha) \tag{31}$$

Let:

$$t^{(\beta-1)(1-\gamma)}E_{\beta,\beta-\gamma(\beta-1)}(t^\beta i^\alpha\omega^\alpha)=g(\omega,t)$$

and

$$F_\alpha^{-1}[g(\omega,t);x]=v(x,t)$$

Then, by using the convolution theorem of the Yang-Fourier transform, we get:

$$u(x,t)=\int_{-\infty}^{+\infty}f(\xi)v(x-\xi,t)(d\xi)^\alpha \tag{32}$$

This completes the proof.

For $0<\beta\leq 1$, $\gamma=1$, the problem (1)-(2) becomes:

$$\begin{cases} {}^{c}\mathrm{D}_t^{\beta} u(x,t) = \dfrac{\partial^{2\alpha}}{\partial x^{2\alpha}} u(x,t) \\ u(x,0) = f(x) \end{cases}$$

Then the solution is:

$$u(x,t) = \int_{-\infty}^{+\infty} f(\xi) v(x-\xi,t)(\mathrm{d}\xi)^{\alpha}$$

where

$$v(x,t) = F_{\alpha}^{-1}[E_{\beta}(t^{\beta} i^{\alpha} \omega^{\alpha}); x]$$

For $0 < \beta \le 1$, $\gamma = 0$, the problem (1)-(2) becomes:

$$\begin{cases} {}^{\mathrm{RL}}\mathrm{D}_t^{\beta} u(x,t) = \dfrac{\partial^{2\alpha}}{\partial x^{2\alpha}} u(x,t) \\ J^{1-\beta} u(x,0) = f(x) \end{cases}$$

Then the solution is:

$$u(x,t) = \int_{-\infty}^{+\infty} f(\xi) v(x-\xi,t)(\mathrm{d}\xi)^{\alpha}$$

where

$$v(x,t) = F_{\alpha}^{-1}[t^{\beta-1} E_{\beta,\beta}(t^{\beta} i^{\alpha} \omega^{\alpha}); x]$$

Conclusion

In this paper, the Yang-Fourier transform and the Laplace transform have been used to solve a new fractional diffusion equations involving the Hilfer fractional derivative and local fractional operator and the exact solution is obtained.

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