NEW HIGH-ORDER CONSERVATIVE DIFFERENCE SCHEME FOR REGULARIZED LONG WAVE EQUATION WITH RICHARDSON EXTRAPOLATION

by

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Numerical solution for the regularized long wave equation is considered by a new three-level conservative implicit finite difference scheme coupled with Richardson extrapolation which has the accuracy of $O(\tau + h^4)$. The scheme is a linear system of equations solved without iteratio. The conservation properties of the algorithm are verified by computing the discrete mass and discrete energy. Existence and uniqueness of the numerical solution are proved. Convergence and stability of the scheme are also derived using energy method. The results of numerical experiments show that our proposed scheme is efficiency.

Key words*: regularized long wave equation, conservative difference scheme, Richardson extrapolation, stability, convergence*

Introduction

Consider the following initial-boundary value problem for the regularized long wave (RLW) equation:

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad (x,t) \in (x_L, x_R) \times (0,T) \tag{1}$$

with an initial condition:

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R] \tag{2}$$

and boundary condition:

$$u(x_L, t) = u(x_R, t) = 0, \quad t \in [0, T] \tag{3}$$

where $u_0(x)$ is a given known function. The RLW equation is originally introduced to describe the behavior of the undular bore by Peregrine [1] and plays a major role in the study of non-linear dispersive waves [2] because of its description a larger number of important physical phenomena, such as shallow water waves and ion acoustic plasma waves.

Mathematical theory for the equation was developed in [3]. Due to non-linear nature of the RLW equation, few exact solutions exist in the literature [4, 5]. Studies mainly consider numerical solution of the problem. These include variational iteratio method [6, 7], finite difference methods [8-15] and various finite element methods such as the Galerkin method [16-20],

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the least squares method [21-23] and collocation method with quadratic B-splines [24], cubic B-splines [25], and recent septic splines [26].

The problem (1)-(3) has two conserved quantities: mass and energy, *i. e.*,

$$Q(t) = \int_{x_L}^{x_R} u(x,t)\mathrm{d}x = \int_{x_L}^{x_R} u_0(x)\mathrm{d}x = Q(0) \tag{4}$$

and

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = \|u_0\|_{L_2}^2 + \|(u_0)_x\|_{L_2}^2 = E(0) \tag{5}$$

where $Q(0)$ and $E(0)$ are two positive constants which relate to the initial condition. Zhang *et al.* [27] pointed out that the conservative difference schemes perform better than the non-conservative ones, and the non-conservative difference schemes may easily show non-linear *blow-up*. Li and Vu-Quoc [28] pointed out that *in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion judge the success of a numerical simulation*. Thus, the purpose of this paper is to present a conservative difference scheme for the initial-boundary value problem (1)-(3). By the Richardson extrapolation, the scheme has the accuracy of $O(\tau^2 + h^4)$ without refined mesh. Moreover, the resulting scheme is a linear system of equations, and it can be solved easily without any iteratios.

Finite difference scheme

Let N, J be any positive integers and $h = (x_R - x_L)/J$ be the step size for the grid such that $x_j = x_L + jh$ ($j = -1, 2,..., J, J + 1$). Let τ be the step for the temporal direction, $t_n = n\tau$ ($n = 0, 1, 2,..., N$), $N = [\mathrm{T}/\tau]$.

Denote $u_j^n \approx u(x_j, t_n)$ and:

$$Z_h^0 = \{u = (u_j) | u_{-1} = u_0 = u_J = u_{J+1} = 0,\ \ j = -1, 0, 1, 2, \cdots, J, J+1\}$$

Define:

$$(u_j^n)_x = \frac{u_{j+1} - u_j^n}{h},\ (u_j^n)_{\bar x} = \frac{u_j^n - u_{j-1}^n}{h},\ (u_j^n)_{\hat x} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

$$(u_j^n)_{\ddot x} = \frac{u_{j+2}^n - u_{j-2}^n}{h},\ (u_j^n)_{\hat t} = \frac{u_j^{n+1} - u_j^{n-1}}{h\tau},\ \bar u_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2h}$$

$$\langle u^n, v^n \rangle = h\sum_{j=1}^{J-1} u_j^n v_j^n,\ \| u^n \|^2 = \langle u^n, v^n \rangle,\ \| u^n \|_\infty = \max_{1\le j\le J-1} | u^n |$$

and in the paper, C denotes a general positive constant which may have different values in different occurrences.

Lemma 1. For a mesh function by Cauchy-Schwarz inequality:

$$\|u_{\ddot x}\|^2 \le \|u_{\hat x}\|^2 \le \|u_x\|^2$$

The following conservative difference scheme for the problem (1)-(3) is considered:

$$(u_j^n)_{\hat t} - \frac{4}{3}(u_j^n)_{x\bar x\hat t} + \frac{1}{3}(u_j^n)_{\hat x\hat x\hat t} + \frac{4}{3}(\bar u_j^n)_{\hat x} - \frac{1}{3}(\bar u_j^n)_{\ddot x} + \frac{4}{9}[u_j^n(\bar u_j^n)_{\hat x} + (u_j^n\bar u_j^n)_{\hat x}]$$

$$-\frac{1}{9}[u_j^n(\bar u_j^n)_{\ddot x} + (u_j^n\bar u_j^n)_{\ddot x}] = 0,\ \ j = 1, 2, \cdots, J-1,\ \ n = 1, 2, \cdots, N-1 \tag{6}$$

$$u_j^0 = u_0(x_j),\ \ j = 0, 1, 2, ..., J \tag{7}$$

$$u_j^1-\frac{4}{3}(u_j^1)_{x\bar{x}}+\frac{1}{3}(u_j^1)_{\hat{x}\hat{x}}=u_0(x_j)-\frac{\partial^2 u_0}{\partial x^2}(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j),\ \ j=1,2,\cdots,J-1 \tag{8}$$

$$u^n\in Z_h^0,\ \ n=0,1,2,\cdots,N \tag{9}$$

Based on the scheme (6)-(9), the discrete versions of (4) and (5) are obtained:

Theorem 1. The scheme (6)-(9) admits the following invariant,

$$Q^n=\frac{h}{2}\sum_{j=1}^{J-1}(u_j^{n+1}+u_j^n)+\frac{2h}{9}\tau\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\hat{x}}-\frac{h}{18}\tau\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\ddot{x}},\ =Q^{n-1}=\cdots=Q^0 \tag{10}$$

$$E^n=\frac{1}{2}\left(\|u^{n+1}\|^2+\frac{4}{3}\|u_x^{n+1}\|^2-\frac{1}{3}\|u_{\hat{x}}^{n+1}\|^2+\|u^n\|^2+\frac{4}{3}\|u_x^n\|^2-\frac{1}{3}\|u_{\hat{x}}^n\|^2\right),\ =E^{n-1}=\cdots=E^0 \tag{11}$$

for n = 1, 2,...,N –1.

Proof. Multiplying (6) with h, then summing up for j from 1 to J – 1, by the boundary condition (9) and formula of summation by parts [29]:

$$h\sum_{j=1}^{J-1}\frac{(u_j^{n+1}-u_j^{n-1})}{2\tau}+\frac{4}{9}h\sum_{j=1}^{J-1}u_j^n(\overline{u}_j^n)_{\hat{x}}-\frac{1}{9}h\sum_{j=1}^{J-1}u_j^n(\overline{u}_j^n)_{\ddot{x}}=0 \tag{12}$$

Again since:

$$h\sum_{j=1}^{J-1}u_j^n(\overline{u}_j^n)_{\hat{x}}=\frac{h}{2}\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\hat{x}}-\frac{h}{2}\sum_{j=1}^{J-1}u_j^{n-1}(u_j^n)_{\hat{x}} \tag{13}$$

$$h\sum_{j=1}^{J-1}u_j^n(\overline{u}_j^n)_{\ddot{x}}=\frac{h}{2}\sum_{j=1}^{J-1}u_j^n(u_j^{n+1})_{\ddot{x}}-\frac{h}{2}\sum_{j=1}^{J-1}u_j^{n-1}(u_j^n)_{\ddot{x}} \tag{14}$$

substitute (13) and (14) into eq. (12), then eq. (10) is obtained.

Taking the inner product of (6) with $2\bar{u}^n$, that is u^{n+1} + n^{n-1}, according to boundary condition (9):

$$\|u^n\|_{\hat{t}}^2+\frac{4}{3}\|u_x^n\|_{\hat{t}}^2-\frac{1}{3}\|u_{\hat{x}}^n\|_{\hat{t}}^2+\frac{8}{3}\langle\overline{u}_{\hat{x}}^n,\overline{u}^n\rangle-\frac{2}{3}\langle\overline{u}_{\ddot{x}}^n,\overline{u}^n\rangle+$$
$$+2\langle\varphi(u_j^n,\overline{u}_j^n),\overline{u}^n\rangle-2\langle\kappa(u_j^n,\overline{u}_j^n),\overline{u}^n\rangle=0 \tag{15}$$

where

$$\varphi(u_j^n,\overline{u}_j^n)=\frac{4}{9}[u_j^n(\overline{u}_j^n)_{\hat{x}}+(u_j^n\overline{u}_j^n)_{\hat{x}}]$$

$$\kappa(u_j^n,\overline{u}_j^n)=\frac{1}{9}[u_j^n(\overline{u}_j^n)_{\ddot{x}}+(u_j^n\overline{u}_j^n)_{\ddot{x}}]$$

Considering

$$\langle\overline{u}_{\hat{x}}^n,\overline{u}^n\rangle=0,\ \langle\overline{u}_{\ddot{x}}^n,\overline{u}^n\rangle=0 \tag{16}$$

$$\langle\varphi(u^n,\overline{u}^n),\overline{u}^n\rangle=0 \tag{17}$$

and

$$\langle\kappa(u^n,\overline{u}^n),\overline{u}^n\rangle=0 \tag{18}$$

Substituting (16)-(18) into eq. (15), we have:

$$\frac{1}{2\tau}\left(\|u^{n+1}\|^2-\|u^{n-1}\|^2\right)+\frac{2}{3\tau}\left(\|u_x^{n+1}\|^2-\|u_x^{n-1}\|^2\right)-\frac{1}{6\tau}\left(\|u_{\hat{x}}^{n+1}\|^2-\|u_{\hat{x}}^{n-1}\|^2\right)=0 \tag{19}$$

By the definition of E^n, (11) is gotten from (19).

Solvability

Next, we are going to prove the solvability of the finite difference scheme (6)-(9).

Theorem 1. The difference scheme (6)-(9) is uniquely solvable.

Proof. Use the mathematical induction. It is obvious that u^0 and $u^1$are uniquely determined by eqs. (7) and (8). Now suppose $u^0, u^1,..., u^{n-1}, u^n$ be solved uniquely. Consider the equation of eq. (6) for u^{n+1}:

$$\frac{1}{2\tau}u_j^{n+1}-\frac{2}{3\tau}(u_j^{n+1})_{x\bar{x}}+\frac{1}{6\tau}(u_j^{n+1})_{\hat{x}\hat{x}}+\frac{2}{3}(u_j^{n+1})_{\hat{x}}-\frac{1}{6}(u_j^{n+1})_{\ddot{x}}+$$
$$+\frac{1}{2}\varphi(u_j^n,u_j^{n+1})-\frac{1}{2}\kappa(u_j^n,u_j^{n+1})=0,\ \ j=1,2,...J-1,\ \ n=1,2,...N-1 \tag{20}$$

Computing the inner product of (20) with u^{n+1}, using (9), we obtain:

$$\frac{1}{2\tau}\left\|u^{n+1}\right\|^2+\frac{2}{3\tau}\left\|u_x^{n+1}\right\|^2-\frac{1}{6\tau}\left\|u_{\hat{x}}^{n+1}\right\|^2+\frac{2}{3}\langle u_{\hat{x}}^{n+1},u^{n+1}\rangle-\frac{1}{6}\langle u_{\ddot{x}}^{n+1},u^{n+1}\rangle$$
$$\frac{1}{2}\left\langle\varphi(u^n,u^{n+1}),u^{n+1}\right\rangle-\frac{1}{2}\left\langle\kappa(u^n,u^{n+1})u^{n+1}\right\rangle=0 \tag{21}$$

Since:

$$\langle u_{\hat{x}}^{n+1},u^{n+1}\rangle=0\quad\langle u_{\ddot{x}}^{n+1},u^{n+1}\rangle=0 \tag{22}$$

$$\langle\varphi(u^n,u^{n+1}),u^{n+1}\rangle=0 \tag{23}$$

and

$$\langle\kappa(u^n,u^{n+1}),u^{n+1}\rangle=0 \tag{24}$$

Substituting (22)-(24) into (21), by *Lemma 1*, we have:

$$\left\|u^{n+1}\right\|^2+\left\|u_x^{n+1}\right\|^2\le 0$$

That is, (20) has only a trivial solution. Therefore, (6) determines u_j^{n+1} uniquely. This completes the proof.

Convergence and stability

Let $v(x, t)$ be the solution of problem (1)-(3) and $v_j^n = u(x_j, t_n)$, the the truncation error of the scheme (6)-(9) is derived:

$$r_j^n=(v_j^n)_{\hat{t}}-\frac{4}{3}(v_j^n)_{x\bar{x}\hat{t}}+\frac{1}{3}(v_j^n)_{\hat{x}\hat{x}\hat{t}}+\frac{4}{3}(\bar{v}_j^n)_{\hat{x}}-\frac{1}{3}(\bar{v}_j^n)_{\ddot{x}}+\varphi(v_j^n,\bar{v}_j^n)-\kappa(v_j^n,\bar{v}_j^n),$$
$$j=1,2,...,J-1,\ \ n=1,2...,N-1 \tag{25}$$

$$v_j=u\ (x_j),\quad j=0,1,2...,J \tag{26}$$

$$v_j^1-\frac{4}{3}(v_j^1)_{x\bar{x}}+\frac{1}{3}(v_j^1)_{\hat{x}\hat{x}}=u_0(x_j)-\frac{\partial^2u_0}{\partial x^2}(x_j)-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j)+r_j^0,$$
$$j=1,2,\cdots,J-1 \tag{27}$$

$$v^n\in Z_h^0,\quad n=0,1,2,\cdots,N \tag{28}$$

According to Taylor expansion, we obtain that:

$$\left|r_j^n\right| = O(\tau^2 + h^4) \tag{29}$$

holds as h, $\tau \to 0$.

For the difference solution of the scheme (6)-(9), we have the following priori estimates.

Lemma 1. Suppose $u_0 \in H_0^1[x_L, x_R]$ then the solution of the initial-boundary value problem (1)-(3) satisfies:

$$\| u \|_{L_2} \le C \quad \| u_x \|_{L_2} \le C \quad \| u \|_{L_\infty} \le C$$

Proof. It follows from (5) that:

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = E(0) = C$$

which yields:

$$\| u \|_{L_2} \le C \quad \| u_x \|_{L_2} \le C$$

By Sobolev inequality, we have:

$$\|u\|_{L_\infty} \le C$$

Lemma 2. Suppose $u_0 \in H_0^1[x_L, x_R]$ then the solution of the scheme (6)-(9) satisfies:

$$\| u^n \| \le C \quad \| u_x^n \| \le C \quad \| u^n \|_\infty \le C$$

for $n = 0, 1, 2,...N$.

Proof. It follows from *Theorem 1* and *Lemma 1* that:

$$\frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u_x^{n+1}\right\|^2 + \left\|u^n\right\|^2 + \left\|u_x^n\right\|^2\right) \le E^n = E^0 = C$$

that is:

$$\| u_n \| \le C \quad \| u_x^n \| \le C$$

By discrete Sobolev inequality [29], we have:

$$\| u^{n|} |_\infty \le C$$

Theorem 1. Suppose $u_0 \in H_0^1[x_L, x_R]$ then the solution u^n of the difference scheme (6)-(9) converges to the solution of the problem (1)-(3) with order $O(\tau^2 + h^4)$ by the $\|\cdot\|_\infty$ norm.

Proof. Letting:

$$e_j^n = v_j^n - u_j^n$$

and subtracting (6)-(9) from (25)-(28), respectively, we have:

$$r_j^n = (e_j^n)_{\hat{t}} - \frac{4}{3}(e_j^n)_{x\bar{x}\hat{t}} + \frac{1}{3}(e_j^n)_{\hat{x}\hat{x}\hat{t}} + \frac{4}{3}(\overline{e}_j^n)_{\hat{x}} - \frac{1}{3}(\overline{e}_j^n)_{\ddot{x}} + \varphi(v_j^n, \overline{v}_j^n) - \varphi(u_j^n, \overline{u}_j^n) -$$
$$-\kappa(v_j^n, \overline{v}_j^n) + \kappa(u_j^n, \overline{u}_j^n), \ \ j = 1, 2, \cdots, J-1, \ \ n = 1, 2, \cdots, N-1 \tag{30}$$

$$e_j^0 = 0, \quad j = 0, 1, 2, \cdots, J \tag{31}$$

$$e_j^1 - \frac{4}{3}(e_j^1)_{x\bar{x}} + \frac{1}{3}(e_j^1)_{\hat{x}\hat{x}} = r_j^0, \ \ j = 1, 2, ...J-1 \tag{32}$$

$$e^n \in Z_h^0, \quad n = 0, 1, 2, \cdots, N \tag{33}$$

Computing the inner product of (32) with e^1, and using the boundary condition (33), we get:

$$\|e^1\|^2 + \frac{4}{3}\|e_x^1\|^2 - \frac{1}{3}\|e_{\hat{x}}^1\|^2 = \langle r^0, e^1 \rangle \tag{34}$$

From (29), Cauchy-Schwarz inequality and *Lemma 1*, we obtain:

$$\|e^1\|^2 + \|e_x^1\|^2 \le O(\tau^2 + h^4)^2 \tag{35}$$

Computing the inner product of (29) with $2\bar{e}^n$ and using (33) again, we have:

$$\begin{aligned}\langle r^n, 2\bar{e}^n \rangle &= \|e^n\|_{\hat{t}}^2 + \frac{4}{3}\|e_x^n\|_{\hat{t}}^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_{\hat{t}}^2 + \frac{8}{3}\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle - \frac{2}{3}\langle \bar{e}_{\ddot{x}}^n, \bar{e}^n \rangle + \\ &+ 2\langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle - 2\langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle\end{aligned} \tag{36}$$

Similarly to (16), we have:

$$\langle \bar{e}_{\hat{x}}^n, \bar{e}^n \rangle = 0, \quad \langle \bar{e}_{\ddot{x}}^n, \bar{e}^n \rangle = 0 \tag{37}$$

According to *Lemma 1*, *Lemma 2*, *Theorem 1* and Cauchy-Schwartz inequality, we get:

$$\begin{aligned}\langle \varphi(v^n, \bar{v}^n) - \varphi(u^n, \bar{u}^n), \bar{e}^n \rangle &= \frac{4}{9}h\sum_{j=1}^{J-1}[e_j^n(\bar{v}_j^n)_{\hat{x}} + u_j^n(\bar{e}_j^n)_{\hat{x}}]\bar{e}_j^n - \frac{4}{9}h\sum_{j=1}^{J-1}(e_j^n\bar{v}_j^n + u_j^n\bar{e}_j^n)(\bar{e}_j^n)_{\hat{x}} \le \\ &\le C\left(\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_{\hat{x}}^n\|^2\right) \le C\left(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2\right)\end{aligned} \tag{38}$$

$$\begin{aligned}\langle \kappa(v^n, \bar{v}^n) - \kappa(u^n, \bar{u}^n), \bar{e}^n \rangle &= \frac{1}{9}h\sum_{j=1}^{J-1}[v_j^n(\bar{v}_j^n)_{\ddot{x}} - u_j^n(\bar{u}_j^n)_{\ddot{x}}]\bar{e}_j^n + \frac{1}{9}h\sum_{j=1}^{J-1}[(v_j^n\bar{v}_j^n)_{\ddot{x}} - (u_j^n\bar{u}_j^n)_{\ddot{x}}]\bar{e}_j^n \le \\ &\le C\left(\|e^n\|^2 + \|\bar{e}^n\|^2 + \|\bar{e}_{\ddot{x}}^n\|^2\right) \le C\left(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2\right)\end{aligned} \tag{39}$$

and

$$\langle r^n, 2\bar{e}^n \rangle = \langle r^n, e^{n+1} + e^{n-1} \rangle \le \| r^n \|^2 + \| e^{n+1} \|^2 + \| e^{n-1} \|^2 \tag{40}$$

Substituting (37)-(40) into (36), we get:

$$\|e^n\|_{\hat{t}}^2 + \frac{4}{3}\|e_x^n\|_{\hat{t}}^2 - \frac{1}{3}\|e_{\hat{x}}^n\|_{\hat{t}}^2 \le \|r^n\|^2 + C\left(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2\right) \tag{41}$$

Letting:

$$B^n = \|e^{n+1}\|^2 + \|e^n\|^2 + \frac{4}{3}\|e_x^{n+1}\|^2 + \frac{4}{3}\|e_x^n\|^2 - \frac{1}{3}\|e_{\hat{x}}^{n+1}\|^2 - \frac{1}{3}\|e_{\hat{x}}^n\|^2$$

and summing up (41) from 1 to n, we have:

$$B^n \le B^0 + C\tau\sum_{l=1}^{n}\|r^l\|^2 + C\tau\sum_{l=0}^{n}\left(\|e^l\|^2 + \|e_x^l\|^2\right) \tag{42}$$

Noticing:

$$\tau\sum_{l=1}^{n}\|r^l\|^2 \le n\tau\max_{1\le l\le n}\|r^l\|^2 \le TO(\tau^2 + h^4)^2$$

From (31) and (35), we have $B_0 = O(\tau^2 + h^4)^2$. Hence, from (42), and *Lemma 1*, we get:

$$\|e^{n+1}\|^2 + \|e^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^n\|^2 \le B^n \le O(\tau^2 + h^4)^2 + C\tau\sum_{l=0}^{n+1}\left(\|e^l\|^2 + \|e_x^l\|^2\right)$$

By discrete Gronwall inequality [29], we have:

$$\|e^n\| \le O(\tau^2 + h^4) \quad \|e_x^n\| \le O(\tau^2 + h^4)$$

Finally, by discrete Sobolev inequality [29], we get:

$$\|e^n\|_\infty \le O(\tau^2 + h^4)$$

This completes the proof of *Theorem 1*.

Similarly, we can prove the stability of the difference solution.

Theorem 2. Under the conditions of *Theorem 1,* the solution of the scheme (6)-(9) is stable by the $\|\cdot\|_\infty$ norm.

Numerical experiments

The single solitary-wave solution of RLW eq. (1) is given by:

$$u(x,t) = A\sec h^2(kx - \omega t + \delta)$$

where

$$A = \frac{3a^2}{1-a^2} \quad k = \frac{a}{2} \quad \omega = \frac{a}{2(1-a^2)}$$

and a and δ are constants.

The scheme (6)-(9) is a linear system of equations which can be solved without iteratio. Take and the initial function of the problem (1)-(3) is re-written:

$$u(x,0) = \sec h^2\left(\frac{1}{4}x\right)$$

In the numerical experiments, we take $x_L = -50$, $x_R = 50$, and $T = 20$. The errors in the sense of L_∞-norm and L_2-norm of the numerical solutions are listed on tab. 1 under different mesh steps τ and h. Table 2 shows that the computational and the theoretical orders of the scheme are very close to each other. Table 3 shows the value of E_n and Q^n at different time. It indicates that the conservation of the scheme (6)-(9) is very good and it is suitable for long-term computation.

Table 1. The errors estimates of numerical solution with various τ and h

	$\tau = 0.2$	$h = 0.01$	$\tau = 0.05$	$h = 0.05$	$\tau = 0.0125$	$h = 0.025$
	$\|\|e^n\|\|$	$\|\|e^n\|\|_\infty$	$\|\|e^n\|\|$	$\|\|e^n\|\|_\infty$	$\|\|e^n\|\|$	$\|\|e^n\|\|_\infty$
$t = 5$	1.293731e-2	6.219891e-3	8.406610e-4	4.019759e-4	5.266221e-5	2.520183e-5
$t = 10$	2.473412e-2	1.122209e-2	1.575704e-3	7.156453e-4	9.871933e-5	4.483281e-5
$t = 15$	3.472503e-2	1.505045e-2	2.206277e-3	9.588093e-4	1.381956e-4	6.004674e-5
$t = 20$	4.338962e-2	1.828892e-2	2.765299e-3	1.166337e-3	1.740699e-4	7.302556e-5

Table 2. The numerical verification of theoretical accuracy $O(\tau^2 + h^4)$

	$\|e^n(h,\tau)\| \big/ \left\|e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\right\|$			$\|e^n(h,\tau)\|_\infty \big/ \left\|e^{4n}\left(\frac{h}{2},\frac{\tau}{4}\right)\right\|_\infty$		
	$\tau = 0.2$ $h = 0.1$	$\tau = 0.05$ $h = 0.05$	$\tau = 0.0125$ $h = 0.025$	$\tau = 0.2$ $h = 0.1$	$\tau = 0.05$ $h = 0.05$	$\tau = 0.0125$ $h = 0.025$
$t = 5$	–	15.389452	15.963269	–	15.473292	15.950260
$t = 10$	–	15.697181	15.961462	–	15.681087	15.962535
$t = 15$	–	15.739194	15.964880	–	15.697027	15.967716
$t = 20$		15.690750	15.976137		15.680645	15.971630

Table 3. Discrete mass and discrete energy with various τ and h

	$\tau = 0.2$	$h = 0.1$	$\tau = 0.05$	$h = 0.05$	$\tau = 0.0125$	$h = 0.025$
	Q^n	E^n	Q^n	E^n	Q^n	E^n
$t = 0$	8.0023652	5.5999999	8.0001481	5.5999999	8.0000090	5.5999999
$t = 5$	8.0023653	5.5999999	8.0001481	5.5999999	8.0000092	5.5999999
$t = 10$	8.0023652	5.5999999	8.0001480	5.5999999	8.0000091	5.5999999
$t = 15$	8.0023623	5.5999999	8.0001451	5.5999999	8.0000062	5.5999999
$t = 20$	8.0022799	5.5999999	8.0000633	5.5999999	8.0000037	5.5999999

From these computational results, it shows that our proposed algorithm is efficient and reliable.

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References

[1] Peregrine, D. H., Long Waves On Beach, *Journal Fluid Mech.*, *27* (1967), 4, pp. 815-827
[2] Bona, J. L., *et al*., Numerical Schemes for a Model of Non-Linear Dispersive Waves, *Journal Comp.* Phys., *60* (1985), 2, pp. 167-96
[3] Benjamin, T. B., *et al.*, Model Equations for Long Waves in Non-Linear Dispersive Systems, *Philos. Trans. Roy. Soc. London A*, *272* (1972), 1220, pp. 47-78
[4] Wazwaz, A. M., Analytic Study on Non-Linear Variants of the RLW and the PHI-Four Equations, *Commun. Non-Linear Sci. Numer. Simulat., 12* (2007), 3, pp. 314-327
[5] Mohammadi, M., Mokhtari, R., Solving the Generalized Regularized Long Wave Equation on the Basis of a Reproducing Kernel Space, *Journal Comput. Appl. Math.*, *235* (2011), 14, pp. 4003-4014
[6] Soliman, A. A., Numerical Simulation of the Generalized Regularized Long Wave Equation by He's Variational Iteratio Method, *Math. Comput. Simulat., 70* (2005), 2, pp. 119-124
[7] Yusufoglu, E., Bekir, A., Application of the Variational Iteratio Method to the Regularized Long Wave Equation, *Comput. Math. Appl*, *54* (2007), 7-8, pp. 1154-1161
[8] Chang, Q., *et al*., Conservative Scheme for a Model of Non-Linear Dispersive Waves and Its Solitary Waves Induced by Boundary Motion, *Journal Comput. Phys*., *93* (1991), 2, pp. 360-375
[9] Kutluay, S., Esen, A., A Finite Difference Solution of the Regularized Long Wave Equation, *Math. Probl. Eng*., *2006* (2006), 1, pp. 1-14
[10] Lin, J., *et al*., High-Order Compact Difference Scheme for the Regularized Long Wave Equation, *Comm. Numer. Methods Engrg*., *23* (2007), 2, pp. 135-156
[11] Cheng, K., *et al*., An Energy Stable Fourth Order Finite Difference Scheme for the Cahn-Hilliard Equation, *Journal Comput. Appl. Math*., *1* (2019), 5, pp. 1-22
[12] Ramos, J. I., Explicit Finite Difference Methods for the EW and RLW Equations, *Appl. Math. Comput.*, *179* (2006), 2, pp. 622-638
[13] Wang, T., *et al*., Conservative Schemes for the Symmetric Regularized Long Wave Equations, *Appl. Math. Comput*., *190* (2007), 2, pp. 1063-1080
[14] Wang, T., Zhang, L., A Conservative Finite Difference Scheme for Generalized Regularized Long-Wave Equation, *Acta Math. Appl. Sin.-E*, *29* (2006), 6, pp. 1091-1098
[15] Zhang, L., Chang, Q., A New Finite Difference Method for Regularized Long Wave Equation, *Journal on Numerical Methods and Computer Application*, *21* (2000), 4, pp. 247-254
[16] Avilez-Valente, P., Seabra-Santos, F. J., A Petrov-Galerkin Finite Element Scheme for the Regularized Long Wave Equation, *Comput. Mech.*, *34* (2004), 4, pp. 256-270
[17] Esen, A., Kutluay, S., Application of a Lumped Galerkin Method to the Regularized Long Wave Equation, *Appl. Math. Comput.*, *174* (2006), 2, pp. 833-845

[18] Guo, L., Chen, H., The H^1-Galerkin Mixed Finite Element Method for the Regularized Long Wave Equation, *Computing*, *77* (2006), 2, pp. 205-221
[19] Mei, L., Chen, Y., Numerical Solutions of RLW Equation Using Galerkin Method with Extrapolation Techniques, *Comput. Phy. Commun.*, *183* (2012), 8, 1609-1616
[20] Saka, B., Dag, I., A Numerical Solution of the RLW Equation by Galerkin Method Using Quartic B-Splines, *Comm. Numer. Methods Engrg.*, *24* (2008), 11, pp. 1339-1361
[21] Dag, I., Least Squares Quadratic B-Spline Finite Element Method for the Regularized Long Wave Equation, *Comput. Methods Appl. Mech. Engrg., 182* (2000), 1, pp. 205-215
[22] Dag, I., Ozer M. N., Approximation of the RLW Equation by the Least Square Cubic B-Spline Finite Element Method, *Appl. Math. Model.*, *25* (2001), 3, pp. 221-231
[23] Gu, H., Chen N., Least-Squares Mixed Finite Element Methods for the RLW Equations, *Numer. Meth. Part. Differ. Equ., 24* (2008), 3, pp. 749-758
[24] Soliman, A. A., Raslan, K. R., Collocation Method Using Quadratic B-Spline for the RLW Equation, *Int. Journal Comput. Math.*, *78* (2001), 3, pp.399-412
[25] Dag, I., *et al*., Application of Cubic B-Splines for Numerical Solution of the RLW Equation, *Appl. Math*. *Comput*., 159 (2004), 2, pp. 373-389
[26] Soliman, A. A., Hussien M. H., Collocation Solution for RLW Equation with Septic Spline, *Appl. Math. Comput*., *161* (2005), 2, pp. 623-636
[27] Zhang, F., *et al*., Numerical Simulation of Non-Linear Schrodinger Systems: A New Conservative Scheme, *Appl. Math. Comput*., *71* (1995), 2-3, pp. 165-177
[28] Li, S., Vu-Quoc, L., Finite Difference Calculus Invariant Structure of a Class of Algorithms for the Non-Linea Klein-Gordon Equation, *SIAM J. Numer. Anal*., *32* (1995), 6, pp. 1839-1875
[29] Zhou, Y., *Application of Discrete Functional Analysis to the Finite Difference Method*, Inter. Acad. Publishers, Beijing, 1990

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