STOCHASTIC TECHNIQUE FOR SOLUTIONS OF NON-LINEAR FIN EQUATION ARISING IN THERMAL EQUILIBRIUM MODEL

by

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In this study, a stochastic numerical technique is used to investigate the numerical solution of heat transfer temperature distribution system using feed forward artificial neural networks. Mathematical model of fin equation is formulated with the help of artificial neural networks. The effect of the heat on a rectangular fin with thermal conductivity and temperature de-pendent internal heat generation is calculated through neural networks optimization with optimizers like active set technique, interior point technique, pattern search, genetic algorithm and a hybrid approach of pattern search - interior point technique, genetic algorithm - active set technique, genetic algorithm - interior point technique, and genetic algorithm - sequential quadratic programming with different selections of weights. The governing fin equation is transformed into an equivalent non-linear second order ODE. For this transformed ODE model we have performed several simulations to provide the justification of better convergence of results. Moreover, the effectiveness of the designed models is validated through a complete statistical analysis. This study reveals the importance of rectangular fins during the heat transformation through the system.

Key words: *heat distribution, thermal conductivity, genetic algorithm, interior point algorithm, active set technique*

Introduction

Heat transfer is the conversion of energy because of temperature differences. When two bodies having different temperatures come in contact, transfer of heat takes place between them until thermal equilibrium is attained [1]. For the consistent working of electronics with high energy density, it is necessary to formulate effective cooling schemes [2]. Free electrons motion within semiconductors causes not only excessive heat loss but also generates the signal noise [3]. Almost every industrial structure is designed to work within definite temperature limits, going beyond these limits by overheating causes system failure. To avoid this problem of excessive heat generation, electronic components have to pass through a complex network of heat resistances [4]. The following different augmentation methods can be used to increase heat transfer. Compound methods, active methods and passive methods [3]. Passive cooling methods are widely preferred for electronic and power devices, as they are low cost, noiseless and easy to use.

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Increase in heat transfer by-passive techniques can be attained by using: treated surfaces, rough surfaces, and extended surfaces (fins). Extended surfaces deliver efficient heat transfer expansion through increasing the surface area. In the extended surface (fins) modern improvements have been made to amend finned surfaces that also tend to improve the heat transfer coefficients [3].

Fins are extended surfaces used in an attempt to increase the heat transfer rate [5]. Both heat conduction and convection take place simultaneously in this case. Convection occurs outside the surface with inside conduction [6]. Recently, many researchers have been contributed in heat transfer, pressure loss and thermal performances of different fins. [7].

Fins are commonly used in many manufacturing and engineering systems such as in air conditioning systems, turbines, electronics apparatus's, and cooling of electronic components [8]. Fins are in different shapes such as rectangular, circular, pin fin rectangular and pin fin triangular, but because of its simple design, low production costs and its convenient constructing process the rectangular fin is broadly used as compare to others [9]. Fins utilization is one of economical way out which can dissipate undesired heat and it has been effectively used for many manufacturing uses. Because of the scientific significance and wide usage of fins, a non-linear differential equation of heat conduction is counted. Aziz and Na [10] considered a non-linear fin equation subject to variable thermal conductivity.

Chung and Iyer [11] examined the optimization difficulties for longitudinal fins. Kraus *et al*. [5] studied many well-known mathematical examples which describe the heat transfer in different types of fins with variable boundary conditions. Khani and Aziz [12] used homotopy method to establish an analytical solution for the thermal performance of a straight fin of trapezoidal profile when both the thermal conductivity and the heat transfer coefficient are temperature dependent. A decomposition technique for solving longitudinal fins was extensively reviewed by Chiu and Chen [13]. Chang [6] reported a decomposition method in order to examine the thermal effects rectangular fins.

Kim *et al.* [14] summarized an estimated solution for fins subject to thermal conductivity. Yang *et al.* [15] used a double decomposition scheme for solving the oscillating base temperature in longitudinal fins. Kundu and Bhania [16] reviewed the optimization study of a constructed fin with thermal conductivity. Aziz and Fang [17] calculated substitute solutions for rectangular fins. Hosseini *et al.* [18] applied homotopy method for the solution of fin subject to internal heat that is temperature dependent. Singla and Das [19] used Adomian decomposition methodology to solve non-linear rectangular fin problem. Duan *et al.* [20] utilized an application of the modified decomposition method for non-linear straight fins.

In the present study, new computational intelligence techniques are presented for solving fins. Artificial intelligence techniques based on neural networks optimized with efficient global and local search methodologies have been extensively used to solve a variety of the linear and non-linear systems based on ODE and PDE [21, 22]. Moreover, 1-D fin problem solution and triangular fins effects have been studied [23], thermal design of a counter flow heat exchanger using air as the working fluid is investigated [24]. Further some latest work has done some one can see the references [25-28].

Transformation function

Consider a fin of rectangular profile, having heat generation, q, thermal conductivity, k, perimeter, P, length, L, and section area, A, which is attached to a surface of constant temperature, T_b. The fin is losing of heat to the surrounding environment, T_0. This loss of heat is through a constant coefficient, h, of convection heat transfer. Here we assumed heat transfer occurred only in the longitudinal x-direction:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(k\frac{\mathrm{d}T}{\mathrm{d}X}\right)-\frac{hP}{A}\left(T-T_0\right)+q=0 \tag{1}$$

$$X=0,\quad \frac{\mathrm{d}T}{\mathrm{d}X}=0,\quad X=L,\quad T=T_b \tag{2}$$

also

$$k=k_0[1+\beta(T-T_0)] \quad \text{and} \quad q=q_0[1+\varepsilon(T-T_0)]$$

where k_0 is the thermal conductivity and q_0 – the internal heat generation when the surrounding temperature is T_0 while ε and β are the parameters signifying heat generation and thermic conductivity variation [18, 19].

Here through variable transformation, we will change the governing equation of the problem.

Considering a new relation:

$$t=\frac{X}{L},\ \theta=\frac{T-T_0}{T_b-T_0} \tag{3}$$

Substituting these values of eq. (3) into eq. (1) we can obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1+\varepsilon_C\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\}-N^2\theta+N^2G\left\{1+\varepsilon_G\theta\right\}=0 \tag{4}$$

$$t=0,\quad \frac{\mathrm{d}\theta}{\mathrm{d}t}=0,\quad t=1,\quad \theta=1 \tag{5}$$

where various dimensionless terms are mentioned:

$$t=\frac{X}{L},\ \varepsilon_C=\beta\left(T_b-T_0\right),\ \theta=\frac{T-T_0}{T_b-T_0},\ N=\sqrt{\frac{L^2hP}{Ak_0}},\ G=\frac{q_0A}{hP\left(T_b-T_0\right)},\ \varepsilon_G=\varepsilon\left(T_b-T_0\right)$$

Mathematical modeling

In this section we presented mathematical modeling of rectangular fin equation, through artificial neural networks (ANN). A mathematical model of the fin equation is formulated with the help of a feed-forward ANN. In the neural network model, the 2nd derivatives can be approximated through applying an activation function (log sigmoid function) [29, 30]:

$$\hat{\theta}(t)=\sum_{i=1}^{N}\alpha_i\left[\frac{1}{1+\mathrm{e}^{-(\eta_i t+\beta_i)}}\right],\ \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t}=\sum_{i=1}^{N}\alpha_i\eta_i\left\{\frac{\mathrm{e}^{-(\eta_i t+\beta_i)}}{\left[1+\mathrm{e}^{-(\eta_i t+\beta_i)}\right]^2}\right\}$$
$$\frac{\mathrm{d}^2\hat{\theta}}{\mathrm{d}t^2}=\sum_{i=1}^{N}\alpha_i\eta_i^2\left\{\frac{2\mathrm{e}^{-2(\eta_i t+\beta_i)}}{\left[1+\mathrm{e}^{-(\eta_i t+\beta_i)}\right]^3}-\frac{\mathrm{e}^{-(\eta_i t+\beta_i)}}{\left[1+\mathrm{e}^{-(\eta_i t+\beta_i)}\right]^2}\right\} \tag{6}$$

where ^ symbol is the estimated values, and α_i, η_i, and β_i are optimization weights or adjustable parameters.

Fitness function

The fitness function, E, for the model has been constructed in an unsupervised manner as the sum of the error due to terms present in the equation and the error in boundary conditions is defined as $E = E_1 + E_2$. The error function related to transformed fin eq. (4) is written:

$$E_1 = \frac{1}{K+1}\sum_{j=0}^{K}\left[\left(1+\varepsilon_c\hat{\theta}_j\right)\left(\hat{\theta}''_j\right)+\varepsilon_c\left(\hat{\theta}'_j\right)-N^2\hat{\theta}_j+N^2G\left(1+\varepsilon_c\hat{\theta}_j\right)\right]^2 \tag{7}$$

where $t \in (t_0 = 0, t_1, t_2,...t_j = K)$ with h step size, where error term E_2 related to the boundary conditions eq. (5) is defined:

$$E_2 = \frac{1}{2}\left[\left(\hat{\theta}'_0\right)^2+\left(\hat{\theta}_1-1\right)^2\right] \tag{8}$$

Learning techniques

The pattern search (PS) technique belongs to the class of optimization algorithms that do not require the gradient of the function under consideration. Hooke and Jeeves [31] were the first to introduce the PS method. However, the convergence of the PS technique was established by Yu [32]. In the PS technique, a sequence of points that approach an optimal point is computed. In each step, the scheme searches a set of points called a mesh around the optimal point of the previous step. The mesh is created by adding the current point to a scalar multiple of a set of vectors called a pattern. If the PS algorithm finds a point in the mesh that improves the objective function at the current point, the new point becomes the current point at the next step of the algorithm.

Optimization steps

Initialization: Adjust the solver according to the necessity of the problem. Initialize the parameter's values in accordance with optimal technique from table of parameter setting as shown in tab. 1. Vector $W = (\alpha_1, \alpha_2,..., \alpha_m, \eta_1, \eta_2,..., \eta_m, \beta_1, \beta_2,..., \beta_m,)$ act as a straight point for solver here m signifies number of neurons. By continuous relations we can estimate second and third order derivative of solution $\theta(t)$.

Table 1. Parameter values for active set technique (AST), PS, and GA

AST		PS		GA	
Parameter	Values	Parameter	Values	Parameter	Values
Start point creation	Randn (1, 30)	Maximum iteration	1000000	Pop init range	(–1, 1)
Maximum perturbation	0.1	Maximum fun evals	10000000	Population size	100000
Minimum perturbation	$1 \cdot 10^{-10}$	Tol fun	$1 \cdot 10^{-35}$	Bounds	(–30, 30)
Maximum iteration	10000	Tol con	$1 \cdot 10^{-31}$	Selection fun	Stoc Uni
Maximum fun evals	15000	Tol mesh	$1 \cdot 10^{-33}$	Generation	1000
Function tolerance	$1 \cdot 10^{-32}$	X tolerance	$1 \cdot 10^{-37}$	Tol fun	$1 \cdot 10^{-28}$
Tol con	$1 \cdot 10^{-31}$	Initial size	1.0	StallGen limit	1000
Fitness limit	$1 \cdot 10^{-37}$	Penalty fact	100	Tol con	$1 \cdot 10^{-27}$
X tolerance	$1 \cdot 10^{-35}$	Bind toler	$1 \cdot 10^{-3}$	Fitness limit	$1 \cdot 10^{-25}$

Fitness evaluation: For optimization built in function in MATLAB is invoked. In optimizationol press start button in order to start the optimization procedure of problem. Now for fitness evaluation of the problem, process starts until the termination criteria achieve.

Termination criteria: If any one of the following criteria achieved, terminate the implementation of the solver:

– Required level of present fitness achieved as shown in tab. 1.
– Total number of iteration accomplished.
– The predefined fitness value is obtained *i. e.* $|E| \leq 10^{-10}$

Storage: Save the optimal weights (variables) and fitness values and computational time taken by the algorithm.

Statistical analysis: Repeat the process from 1-4 from satisfactorily large number of times. A schematic structure of the fin system with different layers is shown in fig. 1.

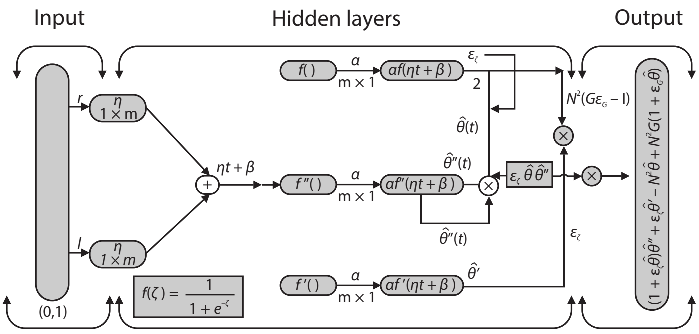

Figure 1. Schematic structure of the fin system

Numerical results

In this section heat transfer analysis of solid fin with rectangular profile having heat generation and air inlet flow is presented using different proposed methodologies. The optimizers are successfully runs with MATLAB R2013(b) software package to find the required numerical results, using Windows XP, with a personal computer having an Intel (R) Core (TM)2 Duo CPU (3.33 GHz) and 2.00 GB RAM (3.19 GHz).

The results are formulated from eqs. (7) and (8) through ANN with setting of 30 weights in which training set is divided into 10 equal space steps between $t \in [0, 1]$ with step size $h = 0.1$. The parameters setting are available in tab. 1 which used in optimtool. Proposed results for different solvers as listed in tab. 2 as $\hat{\theta}_{\text{AST}}$, $\hat{\theta}_{\text{IPT}}$, $\hat{\theta}_{\text{PS}}$, $\hat{\theta}_{\text{GA}}$, and $\hat{\theta}_{\text{PS-AST}}$, yielding the fitness value, $1.6 \cdot 10^{-10}$, $2.42 \cdot 10^{-10}$, $8.92 \cdot 10^{-7}$, $8.74 \cdot 10^{-6}$, $1.92 \cdot 10^{-9}$, respectively. In order to prove the applicability and effectiveness of the proposed scheme a non-linear homogenous fin problem has been considered.

$$\hat{\theta}_{\text{AST}}(t) = \left[\frac{-2.34555}{1+\mathrm{e}^{(1.09481t-2.92787)}}\right] + \left[\frac{1.01497}{1+\mathrm{e}^{(-0.18593t+0.06584)}}\right] + \left[\frac{0.80847}{1+\mathrm{e}^{(-0.30951t-0.12086)}}\right] +$$
$$+ \left[\frac{1.73275}{1+\mathrm{e}^{(-1.03751t+1.61671)}}\right] + \left[\frac{1.98735}{1+\mathrm{e}^{(0.77360t-0.17655)}}\right] + \left[\frac{1.82509}{1+\mathrm{e}^{(1.39539t+2.49811)}}\right] -$$
$$- \left[\frac{2.04728}{1+\mathrm{e}^{(-1.41359t-2.42468)}}\right] + \left[\frac{0.28914}{1+\mathrm{e}^{(0.04972t+0.44847)}}\right] - \left[\frac{0.64901}{1+\mathrm{e}^{(0.18715t-0.23110)}}\right] + \left[\frac{2.85499}{1+\mathrm{e}^{(-1.15206t+2.31949)}}\right] \quad (9)$$

$$\hat{\theta}_{\rm IPT}(t)=\left[\frac{-2.52200}{1+e^{(1.36043t-2.96725)}}\right]+\left[\frac{1.16332}{1+e^{(-0.18239t+0.11170)}}\right]+\left[\frac{0.82965}{1+e^{(-0.43815t-0.11583)}}\right]+$$
$$+\left[\frac{1.80593}{1+e^{(-0.86942t-1.58036)}}\right]+\left[\frac{1.93974}{1+e^{(0.79156t+0.12173)}}\right]+\left[\frac{1.94889}{1+e^{(1.44279t-2.77447)}}\right]-$$
$$-\left[\frac{2.05438}{1+e^{(-1.37172t-2.48089)}}\right]+\left[\frac{0.30926}{1+e^{(0.19066t-0.47224)}}\right]-\left[\frac{0.79343}{1+e^{(0.12093t-0.24818)}}\right]+\left[\frac{2.85761}{1+e^{(-090127t+2.35697)}}\right] \quad (10)$$

$$\hat{\theta}_{\rm GA}(t)=\left[\frac{1.79969}{1+e^{(-0.52723t+0.64758)}}\right]+\left[\frac{0.18581}{1+e^{(2.18903t-1.69836)}}\right]+\left[\frac{0.43426}{1+e^{(-1.17927t+2.15066)}}\right]+$$
$$+\left[\frac{1.15030}{1+e^{(1.38938t+0.48347)}}\right]+\left[\frac{0.82931}{1+e^{(-0.31479t-0.54084)}}\right]+\left[\frac{1.58780}{1+e^{(-0.33341t-0.95770)}}\right]- \quad (11)$$
$$-\left[\frac{1.42205}{1+e^{(0.10474t-1.91373)}}\right]-\left[\frac{0.67637}{1+e^{(0.88349t-2.48799)}}\right]+\left[\frac{0.17410}{1+e^{(-1.80025t-2.50138)}}\right]-\left[\frac{0.40648}{1+e^{(-2.47336t-2.35880)}}\right]$$

Table 2. Reported result with reference (exact) result

t	$\theta(t)$	$\hat{\theta}_{\rm AST}$	$\hat{\theta}_{\rm IPT}$	$\hat{\theta}_{\rm PS}$	$\hat{\theta}_{\rm GA}$	$\hat{\theta}_{\rm PS\text{-}IPT}$
0	0.856922	0.856922	0.856923	0.856739	0.854114	0.856922
0.1	0.858287	0.858288	0.858289	0.858073	0.855269	0.858287
0.2	0.862393	0.862393	0.862394	0.862169	0.859037	0.862393
0.3	0.869261	0.869261	0.869262	0.869045	0.865483	0.869261
0.4	0.878932	0.878932	0.878932	0.878719	0.874707	0.878932
0.5	0.891460	0.89146	0.891461	0.891225	0.886832	0.89146
0.6	0.906917	0.906917	0.906918	0.906635	0.90199	0.906917
0.7	0.925389	0.925389	0.92539	0.92505	0.920296	0.925389
0.8	0.946979	0.946979	0.946979	0.946593	0.941826	0.946979
0.9	0.971805	0.971805	0.971805	0.971401	0.966594	0.971805
1.0	1.000000	1.000000	1.000000	0.999600	0.994540	1.00000

The solutions are given in previous equation remained valid for the entire domain [0, 1]. The values of approximate error (AE) for solution are determined for proposed algorithms and results are provided in tab. 3 for AST, IPT, PS, GA, and PS-ITP lies in the range 10^{-7}-10^{-10}, 10^{-6}-10^{-10}, 10^{-5}-10^{-7}, 10^{-4}-10^{-6}, and 10^{-4}-10^{-6}, respectively. Table 4 shows the comparison of different proposed results of absolute error (AE). The mean value of AE (MAE) is calculated with the interval [0, 1]. Results of MAE for AST, IPT, PS, GA, and PS-IPT are determined, respectively as $4.02\cdot10^{-6}$, $4.04\cdot10^{-7}$, $1{,}65\cdot10^{-4}$, $4.67\cdot10^{-7}$, and $6.93\cdot10^{-3}$. Results of MSE for AST, IPT, PS, GA, and PS-IPT are determined, respectively, as $2.52\cdot10^{-14}$, $2.09\cdot10^{-12}$, $9.99\cdot10^{-8}$, $5.81\cdot10^{-5}$, and $5.81\cdot10^{-5}$. The reliability and self-effectiveness of the solutions provided by stochastic schemes can only be validated through Monte Carlo simulations and its comprehensive statistical analysis. In this regard, 100 independent runs are carried out for each solver as shown in fig. 2.

Table 3. Approximated errors reported by the exact result taken from different algorithms

t	$\lvert \theta_{\text{ref}} - \hat{\theta}_{\text{AST}} \rvert$	$\lvert \theta_{\text{ref}} - \hat{\theta}_{\text{IPT}} \rvert$	$\lvert \theta_{\text{ref}} - \hat{\theta}_{\text{PS}} \rvert$	$\lvert \theta_{\text{ref}} - \hat{\theta}_{\text{GA}} \rvert$	$\lvert \theta_{\text{ref}} - \hat{\theta}_{\text{PS-IPT}} \rvert$
0.0	$9.37 \cdot 10^{-7}$	$3.85 \cdot 10^{-7}$	$3.18 \cdot 10^{-5}$	$1.93 \cdot 10^{-5}$	$1.10 \cdot 10^{-6}$
0.1	$1.38 \cdot 10^{-7}$	$1.13 \cdot 10^{-6}$	$3.14 \cdot 10^{-5}$	$8.74 \cdot 10^{-6}$	$2.96 \cdot 10^{-7}$
0.2	$8.08 \cdot 10^{-7}$	$1.86 \cdot 10^{-7}$	$2.10 \cdot 10^{-5}$	$5.17 \cdot 10^{-5}$	$9.56 \cdot 10^{-7}$
0.3	$4.15 \cdot 10^{-7}$	$2.73 \cdot 10^{-7}$	$7.30 \cdot 10^{-6}$	$6.00 \cdot 10^{-5}$	$5.51 \cdot 10^{-7}$
0.4	$6.47 \cdot 10^{-7}$	$7.71 \cdot 10^{-8}$	$2.64 \cdot 10^{-6}$	$8.50 \cdot 10^{-6}$	$7.43 \cdot 10^{-7}$
0.5	$3.03 \cdot 10^{-7}$	$3.19 \cdot 10^{-7}$	$2.93 \cdot 10^{-6}$	$9.35 \cdot 10^{-5}$	$3.38 \cdot 10^{-7}$
0.6	$3.60 \cdot 10^{-7}$	$2.93 \cdot 10^{-7}$	$4.12 \cdot 10^{-6}$	0.000215	$3.60 \cdot 10^{-7}$
0.7	$7.44 \cdot 10^{-8}$	$4.29 \cdot 10^{-7}$	$1.33 \cdot 10^{-5}$	0.000314	$1.02 \cdot 10^{-7}$
0.8	$5.39 \cdot 10^{-8}$	$1.58 \cdot 10^{-7}$	$1.68 \cdot 10^{-5}$	0.000361	$1.38 \cdot 10^{-7}$
0.9	$2.58 \cdot 10^{-7}$	$2.60 \cdot 10^{-7}$	$1.05 \cdot 10^{-5}$	0.000358	$3.42 \cdot 10^{-7}$
1.0	$1.60 \cdot 10^{-10}$	$2.42 \cdot 10^{-10}$	$8.92 \cdot 10^{-7}$	0.000353	$1.92 \cdot 10^{-7}$

Table 4. Comparison of absolute errors of different algorithms

t	AE (GA-30)	AE (GA-AST)	AE (GA-IPT)	AE (GA-SQP)	AE (GA-45)	AE (GA-60)	AE (PS-IPT)
0	$3.36 \cdot 10^{-3}$	$1.07 \cdot 10^{-5}$	$2.19 \cdot 10^{-6}$	$9.82\ 10^{-6}$	$3.19 \cdot 10^{-3}$	0.006398	$1.10 \cdot 10^{-6}$
0.1	$3.38 \cdot 10^{-3}$	$1.07 \cdot 10^{-5}$	$2.1 \cdot 10^{-6}$	$9.78\text{E} \cdot 10^{-6}$	$3.41 \cdot 10^{-3}$	0.006461	$2.96 \cdot 10^{-7}$
0.2	$3.50 \cdot 10^{-3}$	$9.38 \cdot 10^{-6}$	$1.75 \cdot 10^{-6}$	$8.63 \cdot 10^{-6}$	$3.83 \cdot 10^{-3}$	0.00676	$9.56 \cdot 10^{-7}$
0.3	$3.68 \cdot 10^{-3}$	$7.06 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$	$6.68 \cdot 10^{-6}$	$4.28 \cdot 10^{-3}$	0.007082	$5.51 \cdot 10^{-7}$
0.4	$3.85 \cdot 10^{-3}$	$4.9 \cdot 10^{-6}$	$1.19 \cdot 10^{-6}$	$4.94 \cdot 10^{-6}$	$4.71 \cdot 10^{-3}$	0.007329	$7.43 \cdot 10^{-7}$
0.5	$3.97 \cdot 10^{-3}$	$3.65 \cdot 10^{-6}$	$9.95 \cdot 10^{-7}$	$3.92 \cdot 10^{-6}$	$5.09 \cdot 10^{-3}$	0.007464	$3.38 \cdot 10^{-7}$
0.6	$4.02 \cdot 10^{-3}$	$3.34 \cdot 10^{-6}$	$7.71 \cdot 10^{-7}$	$3.64\ 10^{-6}$	$5.39 \cdot 10^{-3}$	0.007475	$3.60 \cdot 10^{-7}$
0.7	$3.96 \cdot 10^{-3}$	$3.4 \cdot 10^{-6}$	$5.75 \cdot 10^{-7}$	$3.53 \cdot 10^{-6}$	$5.59 \cdot 10^{-3}$	0.00741	$1.02 \cdot 10^{-7}$
0.8	$3.82 \cdot 10^{-3}$	$3.25 \cdot 10^{-6}$	$4.95 \cdot 10^{-7}$	$3.28 \cdot 10^{-6}$	$5.74 \cdot 10^{-3}$	0.007403	$1.38 \cdot 10^{-7}$
0.9	$3.71 \cdot 10^{-3}$	$2.58 \cdot 10^{-6}$	$4.43 \cdot 10^{-7}$	$2.78 \cdot 10^{-6}$	$5.96 \cdot 10^{-3}$	0.007641	$3.42 \cdot 10^{-7}$
1	$3.80 \cdot 10^{-3}$	$1.33 \cdot 10^{-6}$	$1.11 \cdot 10^{-7}$	$1.95 \cdot 10^{-6}$	$6.45 \cdot 10^{-3}$	0.008344	$1.92 \cdot 10^{-9}$

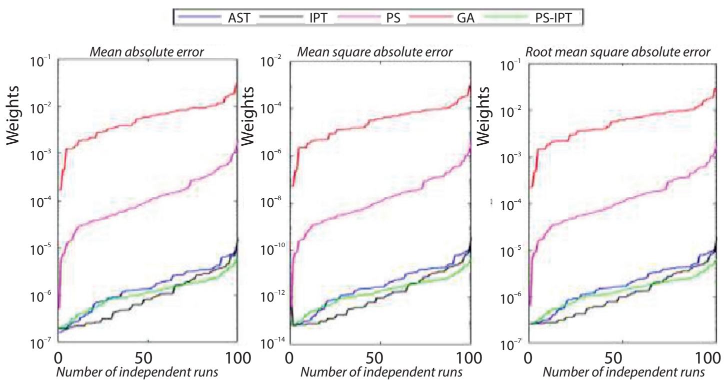

Figure 2. Shows the MAE, MSE, and RMSE taken by 100 independent runs

Statistical analysis

In order to find small differences among results of AE, we presented statistical analysis particularly fitting of normal distribution based on AE of AST, IPT, PS, GA, and hybrid PS-IPT, GA-AST, GA-IPT, and GA-SQP algorithms as shown in fig. 3 for the fin problem. The normal curve fitting technique is used to find how much the normal distribution accurately fits the AE of our proposed results of algorithms in each case of solvers as shown in fig. 3. It also displays the 95% confidence intervals (dotted curves) for the fitted normal distribution. Figure 3 shows that GA-IPT for m = 30 is the best among all algorithms and data is best fitted with normal distribution. Further, fig. 4 shows that GA-IPT for m = 30 is best among all algorithms results and also justifies the normal distribution of data. These confidence levels indicate that the performance of all methods based on the fitted normal distribution and GA-IPT showed high accuracy than the other six in proposed problem. It can be easily observed from these figures that, the result obtained by technique GA-IPT in fin is better than results obtained by others algorithms. It is observed that for m = 30, our techniques show approximately better results from reported results and could potentially minimize the errors.

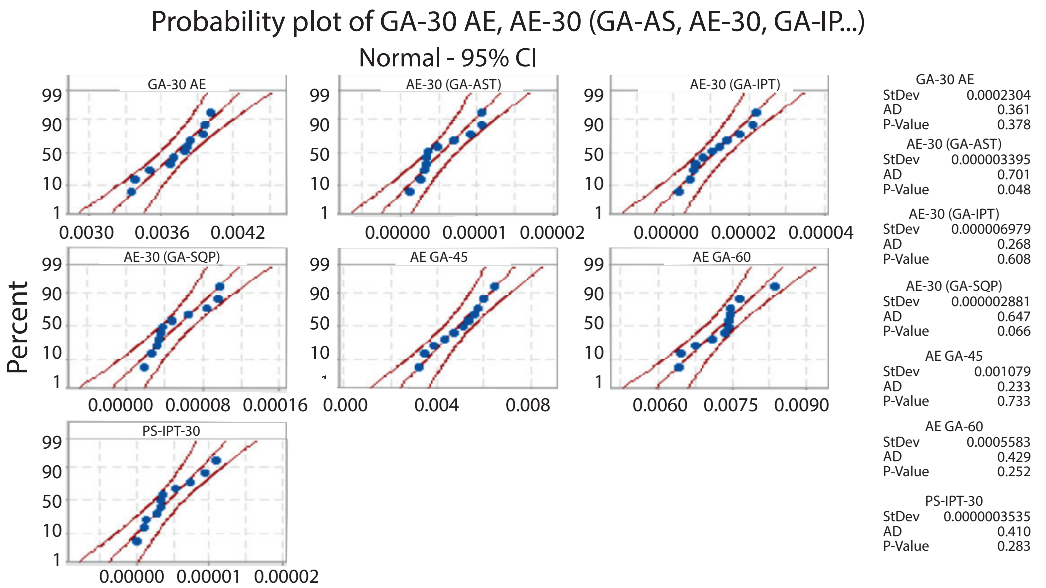

Figure 3. Shows the MAE, MSE, and RMSE taken by 100 independent runs

Conclusion

In this section, we exhibit our findings based on numerical techniques carried out in the previous section. A new artificial intelligence approach is formulated for solving an initial value problem for rectangular fin using feed-forward ANN, AST, IPA, PS, GA, and hybridization PS-IPT, GA-AST, GA-IPT, and GA-SQP. A complete statistical analysis graphical representation is shown in fig. 2. Comparison of the results is carried out with a reference solution calculated through RK-4 method, on the basis of which we have concluded that our presented method has better ability of optimality than others numerical techniques as shown in figs. 4 and 5. The 3-D representation of weights for two solvers, AST and GA, is presented in fig. 6. The accuracy, convergence of the designed unsupervised ANN model for heat transfer temperature

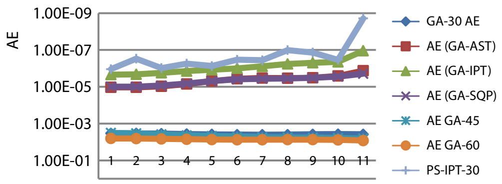

Figure 4. Comparison of AE of different algorithms with different weights

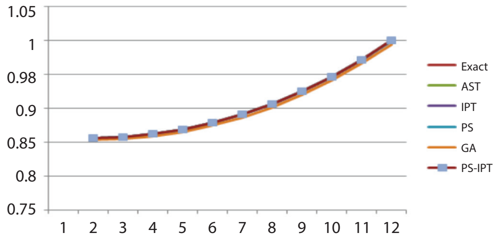

Figure 5. Shows the comparison between the reference and the reported result by different algorithm

distribution fin system can be intensified by use of regenerate competency of intelligent technique based on PS, GA, and its integration with efficient local search technique. Additionally, production of strength of optimization of evolutionary computing through PS hybrid with local search methodologies can be a good alternative for the improve performance for the fin system in term of performance, dependability, intellectual challenges and constancy. In future someone like to use alternate activation functions like tangent sigmoid, radial basis, wavelets hat, mexican hat, *etc.*, for dealing with the heat transfer fin problem with better understanding.

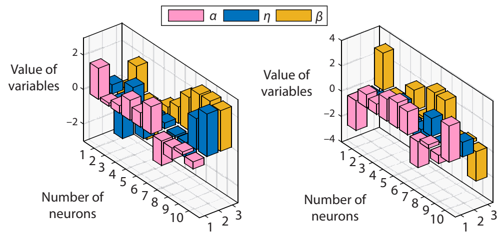

Figure 6. The 3-Dimensional representation of ANN model of weights strained with AST and GA

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