NEW LAPLACE-TYPE INTEGRAL TRANSFORM FOR SOLVING STEADY HEAT TRANSFER PROBLEM

by

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The fundamental purpose of this paper is to propose a new Laplace-type integral transform for solving steady heat transfer problems. The proposed integral transform is a generalization of the Sumudu, and the Laplace transforms and its visualization is more comfortable than the Sumudu transform, the natural transform, and the Elzaki transform. The suggested integral transform is used to solve the steady heat transfer problems, and results are compared with the results of the existing techniques.

Key words: *integral transforms, analytical solutions, heat transfer problems, Laplace-type integral transform*

Introduction

For more than 150 years, the motivation behind integral transforms is easy to understand. The integral transforms have a widely-applicable spirit of converting differential operators into multiplication operators from its original domain into another domain. Besides, the symbolic manipulating and solving the equation in the new domain is easier than manipulation and solution in the original domain of the problem [1-8]. The inverse integral transforms are always used to mapped the manipulated solution back to the original domain to obtain the required result.

In the mathematical literature, the famous classical integral transforms used in differential equations, analysis, theory of functions and integral transforms are the Laplace transform [9] which was first introduced by a French mathematician Pierre-Simon Laplace (1747-1827), the Fourier integral transform [10] devised by another French mathematician Joseph Fourier (1768-1830), and the Mellin integral transform [11] which was introduced by a Finnish mathematician Hjalmar Mellin (1854-1933). Besides, the Laplace transform, the Fourier transform, and the Mellin integral transforms are similar, except in different coordinates and have many applications in science and engineering [12]. Moreover, in mathematics there are many Laplace-types integral transforms such as the Laplace-Carson transform used in the railway engineering [13], the z-transform applied in signal processing [14], the Sumudu transform used in engineering and many real-life problems [15], the Hankel's and Weierstrass transform applied in heat and diffusion equations [16, 17]. In addition, we have the natural transform [18] and Yang transform [19, 20] used in many fields of physical science and engineering.

This paper aims to further introduce a suitable Laplace-type integral transform for solving steady heat transfer problems. We will prove some important theorems and properties

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of the suggested integral transform and illustrated their applications. In the next section, we begin with the definition of the proposed Laplace-type integral transform and introduce some useful theorems of the integral transform.

Definition and theorems

Definition 1. The new Laplace-type integral (NL-TI) transform of the function $v(t)$ of exponential order is defined over the set of functions:

$$A=\left\{v(t):\exists \quad C,\xi_1,\xi_2,>0,\quad |v(t)|<C\mathrm{e}^{\frac{|t|}{\xi_i}},\quad \text{if}\quad t\in(-1)^i[0,\infty)\right\}$$

by the following integral:

$$\Theta[v(t)](s,u)=V(s,u)=u\int_0^\infty \mathrm{e}^{-st}v(ut)\mathrm{d}t=\lim_{\beta\to\infty}\int_0^\beta \mathrm{e}^{(-st)/u}v(t)\mathrm{d}t,\quad s>0,\quad u>0 \tag{1}$$

where Θ is the NL-TI transform operator. It converges if the limit of the integral exists, and diverges if not.

The inverse NL-TI transform is given by:

$$\Theta^{-1}[V(s,u)]=v(t),\quad \text{for}\quad t\geq 0 \tag{2}$$

Equivalently, the complex inversion formula of the NL-TI transform is given by:

$$v(t)=\Theta^{(-1)}[V(s,u)]=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\frac{1}{u}\,\mathrm{e}^{(st/u)}V(s,u)\mathrm{d}s,\quad t>0 \tag{3}$$

where s and u are the NL-TI transform variables, and β is a real constant. The integral in eq. (3) is computed along $s = \beta$ in the complex plane $s = x + iy$.

Theorem 1. The sufficient condition for the existence of the NL-TI transform. If the function $v(t)$ is piecewise continues on every finite interval $[0, t_0]$ and satisfies:

$$|v(t)|\leq C\mathrm{e}^{\beta t} \tag{4}$$

for all $t\in[t_0,\infty)$, and a constant β, then $\Theta[v(t)](s, u)$ exists for all $s/u > \beta$.

Proof. To prove the *Theorem 1*, we must first show that the improper integral converges for $s/u > \beta$. Without loss of generality, we first split the improper integral into two parts namely:

$$\int_0^\infty \mathrm{e}^{(-st/u)}v(t)\mathrm{d}t=\int_0^{t_0}\mathrm{e}^{(-st/u)}v(t)\mathrm{d}t+\int_{t_0}^\infty \mathrm{e}^{(-st/u)}v(t)\mathrm{d}t \tag{5}$$

The first integral on the right hand side of eq. (5) exists by the first hypothesis, hence the existence of the Laplace-type integral transform completely depends on the second integral. Then by the second hypothesis we deduce:

$$\left|\mathrm{e}^{(-st/u)}v(t)\right|\leq C\mathrm{e}^{(-st/u)}\mathrm{e}^{\beta t}=C\mathrm{e}^{-\frac{(s-\beta u)t_0}{u}} \tag{6}$$

Thus:

$$\int_{t_0}^{\infty} C\mathrm{e}^{-\frac{(s-\beta u)t}{u}}\mathrm{d}t = \frac{u}{s-\beta u}C\mathrm{e}^{-\frac{(s-\beta u)t}{u}} \tag{7}$$

Hence, eq. (7) converges for $\beta < s/u$. This implies by the comparison test for improper integrals theorem, $\Theta[v(t)](s, u)$ exists for all $\beta < s/u$. This complete the proof.

In the next theorem, we prove the uniqueness of the NL-TI transform.

Theorem 2. Uniqueness of the NL-TI transform.

Let $v(t)$ and $w(t)$ be continuous functions defined for $t \geq 0$ and having NL-TI transforms of $V(s, u)$ and $W(s, u)$, respectively. If $V(s, u) = W(s, u)$, then $v(t) = w(t)$.

Proof. From the inverse NL-TI transform eq. (3), we have:

$$v(t) = \frac{1}{2\pi i}\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\frac{1}{u}\,\mathrm{e}^{(st/u)}V(s,u)\mathrm{d}s \tag{8}$$

Since $V(s, u) = W(s, u)$ by the second hypothesis, then replacing this in eq. (8), we obtain:

$$v(t) = \frac{1}{2\pi i}\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\frac{1}{u}\,\mathrm{e}^{(st/u)}W(s,u)\mathrm{d}s \tag{9}$$

This implies:

$$v(t) = \frac{1}{2\pi i}\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty}\frac{1}{u}\,\mathrm{e}^{(st/u)}V(s,u)\mathrm{d}s = w(t) \tag{10}$$

Hence, eq. (10) proves the uniqueness of the NL-TI transform.

Theorem 3. Convolution theorem of the NL-TI transform. Let the functions $v(t)$ and $w(t)$ be in set A. If $V(s, u)$ and $W(s, u)$ are the NL-TI transforms of the functions $v(t)$ and $w(t)$, respectively, then:

$$\Theta[(v\ w)(t)] = V(s, u)\ W(s, u) \tag{11}$$

Where $w*v$ is the convolution of two functions $v(t)$ and $w(t)$ which is defined:

$$\int_0^t v(\tau)w(t-\tau)\mathrm{d}\tau = \int_0^t v(t-\tau)w(\tau)\mathrm{d}\tau \tag{12}$$

Proof. Based on eqs. (1) and (12), we get:

$$\left[\Theta\int_0^t v(\tau)w(t-\tau)\mathrm{d}\tau\right] = \int_0^{\infty}\mathrm{e}^{(-st/u)}\left[\int_0^t v(\tau)w(t-\tau)\right]\mathrm{d}\tau$$

Changing the limit of integration yields:

$$\left[\Theta\int_0^t v(\tau)w(t-\tau)\mathrm{d}\tau\right] = \int_0^{\infty}\left[v(\tau)\int_{\tau}^{\infty}\mathrm{e}^{(-st/u)}w(t-\tau)\mathrm{d}t\right]\mathrm{d}\tau$$

Substituting $\vartheta = t - \tau$ in the inner integral, we deduce:

$$\int_{\tau}^{\infty} \mathrm{e}^{(-st/u)} w(t-\tau)\mathrm{d}t = \int_{0}^{\infty} \mathrm{e}^{-[(\vartheta+\tau)/u]s} w(\vartheta)\mathrm{d}\vartheta = \mathrm{e}^{(-\tau s/u)} \int_{0}^{\infty} \mathrm{e}^{(-\vartheta s/u)} w(\vartheta)\mathrm{d}\vartheta = \mathrm{e}^{(-\tau s/u)} W(s,u)$$

Hence:

$$\Theta\left[\int_{0}^{\tau} v(\tau)w(t-\tau)\mathrm{d}\tau\right] = \int_{\tau}^{\infty} v(\tau)\mathrm{e}^{(-st/u)} W(s,u)\mathrm{d}t = W(s,u)\int_{0}^{\infty} v(\tau)\mathrm{e}^{(-st/u)}\mathrm{d}\tau = V(s,u)W(s,u) \tag{13}$$

This complete the proof.

Theorem 4. Derivative of the NL-TI transform. Suppose that $\Theta[v(t)](s, u)$ exists and that $v(t)$ is differentiable n-times on the interval $(0, \infty)$ with n^{th} derivative $v^{(n)}(t)$, then:

$$\Theta[v'(t)](s,u) = \frac{s}{u}V(s,u) - v(0) \tag{14}$$

$$\Theta[v''(t)](s,u) = \frac{s^2}{u^2}V(s,u) - \frac{s}{u}v(0) - v'(0) \tag{15}$$

$$\Theta[v'''(t)](s,u) = \frac{s^3}{u^3}V(s,u) - \frac{s^2}{u^2}v(s,u) - \frac{s}{u}v'(0) - v''(0) \tag{16}$$

$$\vdots$$

$$\Theta[v^{(n)}(t)](s,u) = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)} v^{(k)}(0) \tag{17}$$

Proof. Using *Definition 1* of the NL-TI transform and integration by parts, we deduce:

$$\Theta[v'(t)](s,u) = \int_{0}^{\infty} \mathrm{e}^{(-st/u)} v'(t)\mathrm{d}t = -v(0) + \frac{s}{u}\int_{0}^{\infty} \mathrm{e}^{(-st/u)} v(t)\mathrm{d}t = -v(0) + \frac{s}{u}V(s,u) \tag{18}$$

$$\Theta[v''(t)](s,u) = \frac{s}{u}\Theta[v'(t)](s,u) - v'(0) = \frac{s}{u}\left[-v(0) + \frac{s}{u}V(s,u)\right] - v'(0) =$$

$$= -v'(0) - \frac{s}{u}v(0) + \frac{s^2}{u^2}V(s,u) \tag{19}$$

$$\Theta[v'''(t)](s,u) = \frac{s}{u}\Theta[v''(t)](s,u) - v''(0) = \frac{s}{u}\left[-v'(0) - \frac{s}{u}v(0) + \frac{s^2}{u^2}V(s,u)\right] - v''(0) =$$

$$= -v''(0) - \frac{s}{u}v'(0) - \frac{s^2}{u^2}v(0) + \frac{s^3}{u^3}V(s,u) \tag{20}$$

Finally, eq. (17) follows using mathematical induction.

In the next theorems, we prove the NL-TI transform of Riemann-Liouville fractional derivative ${}^{RL}\mathrm{D}_t^{\alpha}\, v(t)$ [6], and the Caputo fractional derivative ${}^{C}\mathrm{D}_t^{\alpha}\, v(t)$ [6].

Theorem 5. The NL-TI transform of Riemann-Liouville fractional derivative. If $\alpha > 0$, $n = 1 + [\alpha]$ and $v(t)$, $I^{n-\alpha}v(t)$, $(\mathrm{d}/\mathrm{d}t)I^{n-\alpha}v(t)$, …, $(\mathrm{d}^n/\mathrm{d}t^n)I^{n-\alpha}v(t)$, ${}^{RL}\mathrm{D}^{\alpha}v(t) \in A$, then:

$$\Theta\left[{}^{RL}\mathrm{D}_t^{\alpha}\, v(t)\right] = \left(\frac{s}{u}\right)^{\alpha} \Theta[v(t)] \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} I^{n-\alpha} v(0+) \tag{21}$$

where I^{α} is the Riemann-Liouville fractional integral.

Proof. Since ${}^{RL}\mathrm{D}_t^{\alpha}\, v(t) = (\mathrm{d}^n/\mathrm{d}t^n)I^{n-\alpha}v(t)$. Let $g(t) = I^{n-\alpha}v(t)$, then ${}^{RL}\mathrm{D}_t^{\alpha}\, v(t) = (\mathrm{d}^n/\mathrm{d}t^n)v(t)$. Applying the hypothesis of *Theorem 4*, we get:

$$\Theta\left[{}^{RL}\mathrm{D}_t^{\alpha}\, v(t)\right] = \left(\frac{s}{u}\right)^{n} \Theta[v(t)] - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1} v^{(k)}(0+) =$$
$$= \left(\frac{s}{u}\right)^{\alpha} \Theta[v(t)] - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-k-1} \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} I^{n-\alpha} v(0+)$$

The proof ends.

Theorem 6. The NL-TI transform of Caputo fractional derivative. Assume $\alpha > 0$, $n = 1 + [\alpha]$, and:

$$v(t),\quad \frac{\mathrm{d}}{\mathrm{d}t}v(t),\quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t),\quad \ldots,\quad \frac{\mathrm{d}^n}{\mathrm{d}t^n}v(t),\quad {}^{C}\mathrm{D}^{\alpha}\, v(t) \in A, \text{ then:}$$

$$\Theta\left[{}^{C}\mathrm{D}_t^{\alpha}\, v(t)\right] = \left(\frac{s}{u}\right)^{\alpha} \Theta[v(t)] - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-k-1} v^{(k)}(0+) \tag{22}$$

where ${}^{C}\mathrm{D}_t^{\alpha}$ is the Caputo fractional derivative.

Proof. Applying the Caputo fractional derivative [6] and *Theorem 3*, we deduce:

$${}^{C}\mathrm{D}_t^{\alpha}\, v(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} v^{(n)}(\tau)\mathrm{d}\tau = \frac{1}{n-\alpha}\tau^{n-\alpha-1} v^{(n)}(\tau)$$

Finally, using the hypothesis of *Theorem 4* yields:

$$\Theta\left[{}^{C}\mathrm{D}_t^{\alpha}\, v(t)\right] = \frac{1}{\Gamma(n-\alpha)}\Theta[t^{n-\alpha-1}]\Theta[v^{(n)}(\tau)] = \left(\frac{s}{u}\right)^{\alpha}\Theta[v(\tau)] - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-k-1} v^{(k)}(0+)$$

This completes the proof.

Some properties of the NL-TI transform

Property 1. Linearity property of the NL-TI transform. Let $\Theta[v(t)](s, u) = V(s, u)$ and $\Theta[w(t)](s, u) = W(s, u)$, then:

$$\Theta[\alpha v(t) + \beta w(t)](s, u) = \alpha\Theta[v(t)](s, u) + \beta\Theta[w(t)](s, u) \tag{23}$$

where α and β are constants.

Proof. Linearity property follows directly from *Definition 1*.

Property 2. Exponential shifting property of the NL-TI transform. Let the function $v(t) \in A$ and α is an arbitrary constant, then:

$$\Theta[\mathrm{e}^{\alpha t}v(t)](s, u) = V(s - \alpha u, u) \tag{24}$$

Proof. Using *Definition 1* of the NL-TI transform, we get:

$$\Theta[v(t)](s,u) = u\int_0^\infty \mathrm{e}^{-st}v(ut)\,\mathrm{d}t \tag{25}$$

Then:

$$\Theta[\mathrm{e}^{\alpha t}v(t)](s,u) = \int_0^\infty \mathrm{e}^{\alpha t}\mathrm{e}^{(-st/u)}v(t)\mathrm{d}t = \int_0^\infty \mathrm{e}^{[-(s-\alpha u)/u)]}v(t)\mathrm{d}t =$$

$$= u\int_0^\infty \mathrm{e}^{-(s-\alpha u)t}v(ut)\mathrm{d}t = \Theta[v(t)](s-\alpha u, u) = V(s-\alpha u, u) \tag{26}$$

In particular:

$$\Theta\left[\sin(3t)\mathrm{e}^{(-4t)}\right](s,u) = \Theta[\sin(3t)][s-(-4u),u] = \Theta[\sin(3t)](s+4u,u) \tag{27}$$

Based on *Definition 1*, the NL-TI transform of $\sin(3t)$ is given by:

$$\Theta[\sin(3t)](s,u) = \int_0^\infty \mathrm{e}^{-(st/u)}\sin(3t)\mathrm{d}t = \frac{1}{2i}\left(\frac{u}{s-3iu} - \frac{u}{s+3iu}\right) = \frac{3u^2}{s^2+9u^2} \tag{28}$$

So, replacing the variable s with $(s + 4u)$ in eq. (28), we obtain:

$$\Theta[\sin(3t)](s+4u,u) = \frac{3u^2}{(s+4u)^2+9u^2} = \frac{3u^2}{s^2+8us+25u^2} \tag{29}$$

Alternatively:

$$\int_0^\infty \sin(3t)\mathrm{e}^{-4t}\mathrm{e}^{-(st/u)}\mathrm{d}t = \frac{3u^2}{s^2+8us+25u^2} \tag{30}$$

Moreover:

$$\Theta(t\mathrm{e}^{\alpha t})(s,u) = \frac{u^2}{(s-\alpha u)^2} = \begin{cases} \dfrac{1}{(s-\alpha)^2}, & u=1,\ \text{Laplace transform [6]} \\ \dfrac{u^2}{(1-\alpha u)^2}, & S=1,\ \text{Yang transform [19]} \end{cases} \tag{31}$$

This complete the proof.

Property 3. New Laplace-type transform of integral. Let $\Theta[v(t)] = V(s, u)$ and $v(t) \in A$, then:

$$\Theta\left[\int_0^t v(\zeta)\mathrm{d}\zeta\right] = \frac{u}{s}V(s,u) \tag{32}$$

Proof. Let $w(t) = \int_0^t v(\zeta)\mathrm{d}\zeta$, then $w'(t) = v(t)$ and $w(0) = 0$. Computing the NL-TI transform of both sides, we get:

$$\Theta[w'(t)](s,u) = \Theta[v(t)](s,u) = \frac{s}{u}\Theta[w(t)](s,u) - w(0) = V(s,u) \tag{33}$$

This implies:

$$\Theta\left[\int_0^t v(\zeta)\mathrm{d}\zeta\right] = \frac{u}{s}V(s,u) \tag{34}$$

This completes the proof.

Property 4. Multiple shift property of the NL-TI transform. Let $\Theta[v(t)](s, u) = V(s, u)$ and $v(t) \in A$, then:

$$\Theta[t^n v(t)](s,u) = (-u)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}V(s,u) \tag{35}$$

Proof. By *Definition 1* of the NL-TI transform and Leibniz's rule, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s}V(s,u) = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-(st/u)}v(t)\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s}[\mathrm{e}^{-(st/u)}]v(t)\mathrm{d}t = -\frac{1}{u}\int_0^\infty \mathrm{e}^{-(st/u)}tv(t)\mathrm{d}t =$$

$$\Theta[tv(t)](s,u) = -u\frac{\mathrm{d}}{\mathrm{d}s}V(s,u) \tag{36}$$

Thus, eq. (36) proves the theorem for $n = 1$. To generalized the theorem, we apply the induction hypothesis. Let assume the theorem holds for $n = k$ that is:

$$\int_0^\infty \mathrm{e}^{-(st/u)}t^k v(t)\mathrm{d}t = (-u)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k}V(s,u) \tag{37}$$

Then:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-(st/u)}t^k v(t)\mathrm{d}t = (-u)^k \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}}V(s,u) \tag{38}$$

Alternatively, using Leibniz's rule, we deduce:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-(st/u)}t^k v(t)\mathrm{d}t = \int_0^\infty \frac{\partial}{\partial s}\mathrm{e}^{-(st/u)}t^k v(t)\mathrm{d}t = -\frac{1}{u}\int_0^\infty \mathrm{e}^{-(st/u)}t^{k+1}v(t)\mathrm{d}t = (-u)^k \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}}V(s,u) \tag{39}$$

This implies:

$$\int_0^{\infty} \mathrm{e}^{-(st/u)} t^{k+1} v(t)\mathrm{d}t = (-u)^{k+1} \frac{\mathrm{d}^{k+1}}{\mathrm{d}s^{k+1}} V(s,u) \tag{40}$$

Since, eq. (40) holds for $n = k + 1$, then by induction hypothesis the prove is complete.

Applications

In this section, we illustrate the applicability of the proposed Laplace-type integral transform on steady heat transfer problems to proves its efficiency and high accuracy.

Example 1. Consider the following steady heat transfer problem:

$$-hMv(t) = \rho \Lambda_{c_p} v'(t) \tag{41}$$

subject to the initial condition:

$$v(0) = \beta \tag{42}$$

where h is the convection heat transfer coefficient, M – the surface area of the body, ρ – the density of the body, Λ – the volume, c_p – the specific heat of the material, and $v(t)$ – the temperature.

Applying the NL-TI transform on both sides of eq. (41), we get:

$$-hMV(s,u) = \rho \Lambda_{c_p} \left[\frac{s}{u} V(s,u) - v(0) \right] \tag{43}$$

Substituting the given initial condition and simplifying, we get:

$$V(s,u) = \frac{\beta u}{s + \dfrac{hM}{\rho \Lambda_{c_p}} u} \tag{44}$$

Taking the inverse NL-TI transform of eq. (44), we get:

$$v(t) = \beta \mathrm{e}^{-\frac{hM}{\rho \Lambda_{c_p}} t} \tag{45}$$

The exact solution is in excellent agreement with the result obtained in [5, 20].

Example 2. Consider the following steady heat transfer problem:

$$v_t(x, t) = 2v_{xx}(x, t), \quad 0 < x < 5, \quad t > 0 \tag{46}$$

Subject to the boundary and initial conditions:

$$v(0, t) = 0, \quad v(5, t) = 0, \quad v(x, 0) = 10 \sin(4\pi x) - 5\sin(6\pi x) \tag{47}$$

Applying the NL-TI transform on both sides of eq. (46), we deduce:

$$\frac{s}{u} V(x,s,u) - v(x,0) = \frac{2\mathrm{d}^2 V(x,s,u)}{\mathrm{d}x^2} \tag{48}$$

Substituting the given initial condition and simplifying, we get:

$$\frac{2\mathrm{d}^2V(x,s,u)}{\mathrm{d}x^2}-\frac{s}{u}V(x,s,u)=-10\sin(4\pi x)+5\sin(6\pi x) \tag{49}$$

The general solution of eq. (49) can be written:

$$V(x, s, u) = V_h(x, s, u) + V_p(x, s, u) \tag{50}$$

where $V_h(x, s, u)$ is the solution of the homogeneous part which is given:

$$V_h(x,s,u)=\alpha_1\mathrm{e}^{\sqrt{(s/u)x}}+\alpha_2\mathrm{e}^{-\sqrt{(s/u)x}} \tag{51}$$

and $V_p(x, s, u)$ is the solution of the inhomogeneous part which is given by:

$$V_p(x, s, u) = \alpha\sin(4\pi x) - \beta\sin(6\pi x) \tag{52}$$

Applying the boundary conditions on eq. (51), yields:

$$\alpha\ +\alpha_2=0 \ \Rightarrow\ \alpha_1\mathrm{e}^{\sqrt{(s/u)x}}+\alpha_2\mathrm{e}^{-\sqrt{(s/u)x}}=0\ \Rightarrow\ V_h(x,s,u)=0 \tag{53}$$

since $\alpha_1 = \alpha_2 = 0$.

Using the method of undetermined coefficients on the inhomogeneous part, we get:

$$V_p(x,s,u)=10\sin(4\pi x)\frac{u}{s+32\pi^2u}-5\sin(6\pi x)\frac{u}{s+72\pi^2u'} \tag{54}$$

since $\alpha = 10[u/(s + 32\pi^2u')]$ and $\beta = -5[u/(s + 72\pi^2u)]$.

Then eq. (50) will become:

$$V(x,s,u)=10\sin(4\pi x)\frac{u}{s+32\pi^2u}-5\sin(6\pi x)\frac{u}{s+72\pi^2u} \tag{55}$$

Taking the inverse NL-TI transform of eq. (55), we obtain:

$$v(x,t)=10\mathrm{e}^{-32\pi^2t}\sin(4\pi x)-5\mathrm{e}^{-72\pi^2t}\sin(6\pi x) \tag{56}$$

The exact solution is the same with the result obtained in [9].

Example 3. Consider the following fractional porous medium equation:

$$\mathrm{D}_t^\alpha v(x,t)=\mathrm{D}_x[v(x,t)\mathrm{D}_x v(x,t)],\quad 0<\alpha\le 1 \tag{57}$$

subject to the initial condition:

$$v(x, 0) = x \tag{58}$$

Applying *Theorem 6* on eq. (57) subject to the initial condition, we obtain:

$$\Theta[v(x,t)]=\frac{u}{s}x+\frac{u^\alpha}{s^\alpha}\Theta\{\mathrm{D}_x[v(x,t)\mathrm{D}_x v(x,t)]\} \tag{59}$$

Computing the inverse NL-TI transform on both sides of eq. (59), we deduce:

$$v(x,t)=x+\Theta^{-1}\left(\frac{u^\alpha}{s^\alpha}\Theta\{\mathrm{D}_x[v(x,t)\mathrm{D}_x v(x,t)]\}\right) \tag{60}$$

Based on the basic idea of the HAM, see [6] and references therein, we have:

$$v(x,t)=\sum_{n=0}^{\infty}p^n v_n(x,t) \tag{61}$$

Then eq. (60) will become:

$$\sum_{n=0}^{\infty}p^n v_n(x,t)=x+p\left(\Theta^{-1}\left\{\frac{u^\alpha}{s^\alpha}\Theta\left[\mathrm{D}_x\left(\sum_{n=0}^{\infty}p^n H_n\right)\right]\right\}\right) \tag{62}$$

where H_n is the He's polynomials [6] which represent the non-linear terms $v(x, t)D_x v(x, t)$.

Some few components of the non-linear terms H_n are computed:

$$H_0 = v_0 v_{0x},\quad H_1 = v_0 v_{1x} + v_1 v_{0x},\quad H_2 = v_{0x} v_2 + v_{1x} v_1 + v_{1x} v_0,\ \dots$$

On comparing the coefficients of same powers of p in eq. (62), we get the following approximations:

$$p^0: v_0(x,t)=x$$

$$p^1: v_1(x,t)=\Theta^{-1}\left\{\frac{u^\alpha}{s^\alpha}\Theta[\mathrm{D}_x(H_0)]\right\}=\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$p^2: v_2(x,t)=\Theta^{-1}\left\{\frac{u^\alpha}{s^\alpha}\Theta[\mathrm{D}_x(H_1)]\right\}=0$$

$$\vdots$$

$$p^n: v_n(x,t)=\Theta^{-1}\left\{\frac{u^\alpha}{s^\alpha}\Theta[\mathrm{D}_x(H_{n-1})]\right\}=0,\quad \text{for}\quad n\geq 2$$

Then the solution of eqs. (57) and (58) is given by:

$$v(x,t)=x+\frac{t^\alpha}{\Gamma(1+\alpha)} \tag{63}$$

The result obtained in eq. (63) is in excellent agreement with the result obtained in [6]. The special case of eq. (63) when $\alpha = 1$ is given by:

$$v(x, t) = x + t \tag{64}$$

The result of eq. (64) is in closed agreement with the result obtained in [6, 7].

Conclusion

In this paper, we introduced a powerful Laplace-type integral transform for finding a solution of steady heat transfer problems. The proposed Laplace-type integral transform converges to both Yang transform, and the Laplace transforms just by changing variables. Many interesting properties of the suggested integral transform are discussed and successfully applied to steady heat transfer problems. Finally, based on the efficiency and simplicity of the

Laplace-type integral transform, we conclude that it is a powerful mathematical tool for solving many problems in science and engineering.

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Nomenclature

c_p – convection of heat transfer coefficient, $[\mathrm{Jkg^{-1}K^{-1}}]$
h – convection heat transfer coefficient, $[\mathrm{Wm^{-2}K^{-1}}]$
x, t – space co-ordinates, [m]
$v(t)$ – temperature, [K]
$v(x, t)$ – temperature, [K]

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