From the Guest Editors

The mathematical models for the heat and fluid flow based on the Newton-Leibniz calculus have been successfully developed to describe the transport processes of the heat and mass transfer in fluid, solid and gases. Since the nature is fractal, there exists a great many of the mathematical models for them based on the other calculi due to the limitation of the classical models for them. The calculi conclude the derivatives and integrals as follows [1, 2].

In fact, there is the singular kernel in the formulae of the classical fractional derivative and integral.

The Riemann-Liouville fractional derivative is defined as follows [3, 4]:

$$ {}_{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha} j(t) = \frac{1}{\Gamma(\kappa-\alpha)} \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_a^t \frac{j(\tau)}{(\tau-t)^{\alpha-\kappa+1}} \mathrm{d}\tau \tag{1} $$

The Liouville-Caputo fractional derivative is defined as follows [3, 5]:

$$ {}_{\mathrm{LC}}\mathrm{D}_{a+}^{\alpha} j(t) = \frac{1}{\Gamma(\kappa-\alpha)} \int_a^t \frac{j^{(\kappa)}(\tau)}{(\tau-t)^{\alpha-\kappa+1}} \mathrm{d}\tau \tag{2} $$

The Riemann-Liouville fractional integral is defined as follows [3, 4]:

$$ {}_{\mathrm{RL}}I_{a+}^{\alpha} j(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{j(\tau)}{(\tau-t)^{1-\alpha}} \mathrm{d}\tau \tag{3} $$

In general, eqs. (1) and (2) can be written as follows [6]:

$$ {}_{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha} j(t) = \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_a^t \Pi(\tau-t)\, j(\tau)\, \mathrm{d}\tau \tag{4} $$

and

$$ {}_{\mathrm{LC}}\mathrm{D}_{a+}^{\alpha} j(t) = \int_a^t \Pi(\tau-t)\, j^{(\kappa)}(\tau)\, \mathrm{d}\tau \tag{5} $$

respectively, where $\Pi(\tau - t) = 1/[\Gamma(\kappa - \alpha)(\tau - t)^{\alpha - \kappa + 1}]$

For more details of the classical fractional derivative and integral, see [1]. There exist the weakly singular and non-singular kernels in the formulae of the fractional derivative and integral.

For the weakly singular kernel, the fractional derivative within weakly singular kernel is defined by[1]:

$$ {}_{\mathrm{RL}}^{*}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_a^t \frac{\mathrm{e}^{-\lambda(\tau-t)}}{(\tau-t)^{\alpha-\kappa+1}} f(\tau)\, \mathrm{d}\tau \tag{6} $$

The fractional derivative within weakly singular kernel is defined by [1]:

$$ {}_{\mathrm{LC}}^{*}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \int_a^t \frac{\mathrm{e}^{-\lambda(\tau-t)}}{(\tau-t)^{\alpha-\kappa+1}} f^{(\kappa)}(\tau)\, \mathrm{d}\tau \tag{7} $$

As the special cases, we have [1]:

$$ {}_{\mathrm{RL}}^{\ *}\mathrm{D}_{\lambda,+}^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_{-\infty}^{t} \frac{\mathrm{e}^{-\lambda(\tau-t)}}{(\tau-t)^{\alpha-\kappa+1}} f(\tau)\,\mathrm{d}\tau \tag{8} $$

and [1]:

$$ {}_{\mathrm{LC}}^{\ *}\mathrm{D}_{\lambda,+}^{\alpha} f(t) = \frac{1}{\Gamma(\kappa-\alpha)} \int_{-\infty}^{t} \frac{\mathrm{e}^{-\lambda(\tau-t)}}{(\tau-t)^{\alpha-\kappa+1}} f^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{9} $$

Similarly, we get:

$$ {}_{\mathrm{RL}}^{\ *}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_{a}^{t} \Lambda_{\lambda}(\tau-t) f(\tau)\,\mathrm{d}\tau \tag{10} $$

and

$$ {}_{\mathrm{LC}}^{\ *}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \int_{a}^{t} \Lambda_{\lambda}(\tau-t) f^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{11} $$

where $\Lambda_{\lambda}(\tau - t) = \mathrm{e}^{-\lambda(\tau - t)}/[\Gamma(\kappa - \alpha)(\tau - t)^{\alpha - \kappa + 1}]$.

For the non-singular kernels, we have:

$$ {}_{\mathrm{RL}}^{\ **}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_{a}^{t} \Omega_{\lambda}(\tau-t) f(\tau)\,\mathrm{d}\tau \tag{12} $$

and

$$ {}_{\mathrm{LC}}^{\ **}\mathrm{D}_{\lambda,a+}^{\alpha} f(t) = \int_{a}^{t} \Omega_{\lambda}(\tau-t) f^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{13} $$

where $\Omega_{\lambda}(\tau - t)$ is the special functions with the premaster λ.

As a typical example, the general fractional derivative involving the Miller-Ross kernel is defined as follows [1]:

$$ {}_{\mathrm{MR}}^{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_{a}^{t} \mathfrak{M}_{\alpha}\left[-\lambda(t-\tau)^{\alpha}\right] j(\tau)\,\mathrm{d}\tau \tag{14} $$

where the Miller-Ross function is [1]:

$$ \mathfrak{M}_{\alpha}\left(\lambda t^{\alpha}\right) = \sum_{\kappa=0}^{\infty} \frac{\lambda^{\kappa} t^{\kappa+\alpha}}{\Gamma(\kappa+1+\nu)} $$

The general fractional derivative involving the Miller-Ross kernel is defined as follows [1]:

$$ {}_{\mathrm{MR}}^{\mathrm{LS}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \int_{a}^{t} \mathfrak{M}_{\alpha}\left[-\lambda(t-\tau)^{\alpha}\right] j^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{15} $$

The general fractional derivative involving the Lorenzo-Hartley kernel is defined [1]:

$$ {}_{\mathrm{MR}}^{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \frac{\mathrm{d}^{\kappa}}{\mathrm{d}t^{\kappa}} \int_{a}^{t} F_{\alpha}\left[-\lambda(t-\tau)^{\alpha}\right] j(\tau)\,\mathrm{d}\tau \tag{16} $$

where the Lorenzo-Hartley function is [1]:

$$ F_{\alpha}\left(\lambda t^{\alpha}\right) = \sum_{\kappa=0}^{\infty} \frac{\lambda^{\kappa} t^{(\kappa+1)\alpha-1}}{\Gamma\left[(\kappa+1)\alpha\right]} $$

The general fractional derivative involving the Lorenzo-Hartley kernel is defined:

$$ {}_{\mathrm{MR}}^{\mathrm{LS}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \int_a^t F_\alpha \left[-\lambda (t-\tau)^\alpha \right] j^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{17} $$

The general fractional derivative involving the Gorenflo-Mainardi kernel is defined [1]:

$$ {}_{\mathrm{MR}}^{\mathrm{RL}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \frac{\mathrm{d}^\kappa}{\mathrm{d}t^\kappa} \int_a^t G_{\alpha,\gamma} \left[-\lambda (t-\tau)^\alpha \right] j(\tau)\,\mathrm{d}\tau \tag{18} $$

where the Gorenflo-Mainardi function is [1]:

$$ G_{\alpha,\gamma}\left(\lambda t^\alpha\right) = \sum_{\kappa=0}^{\infty} \frac{\lambda^\kappa t^{\kappa\alpha-1+\gamma}}{\Gamma(\kappa\alpha+\gamma)} $$

The general fractional derivative involving the Gorenflo-Mainardi kernel is defined [1]:

$$ {}_{\mathrm{MR}}^{\mathrm{LS}}\mathrm{D}_{a+}^{\alpha,\kappa,\lambda} j(t) = \int_a^t G_{\alpha,\gamma} \left[-\lambda (t-\tau)^\alpha \right] j^{(\kappa)}(\tau)\,\mathrm{d}\tau \tag{19} $$

The local fractional derivative at $x = x_0$ is defined [7-9]:

$$ \mathrm{D}^{(\alpha)} f(x) = \lim_{x \to x_0} \frac{\Delta^\alpha \left[f(x) - f(x_0) \right]}{(x - x_0)^\alpha} \tag{20} $$

where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1 + \alpha)\Delta[f(x) - f(x_0)]$.

The local fractional integral is defined as follows [7-9]:

$$ {}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)(\mathrm{d}x)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta x_k \to 0} \sum_{k=0}^{N-1} f(x_k)(\Delta x_k)^\alpha \tag{21} $$

where $\Delta x_k = x_{k+1} - x_k$ with $x_0 = a < x_1 < \cdots < x_{N-1} < x_N = b$.

Based on the general calculus, fractional calculus, local fractional calculus (also called the fractal calculus), and general fractional calculus, there are the open problems for the mathematical models for the heat and fluid flow. In our special issue, we had collected more than 140 papers, and selected 46 papers for publication. I would like to express thanks for Prof. Dr. Simeon Oka and Dr. Vukman Bakić to support and help to publish the special issue.

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