THE $\bar{\lambda}$-STATISTICALLY CONVERGENT DOUBLE SEQUENCES IN FUZZY NORMED SPACES

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Original scientific paper
https://doi.org/10.2298/TSCI180930339K

We introduce double $\bar{\lambda}$-statistically convergent sequences and double $\bar{\lambda}$-statistically Cauchy sequences in the fuzzy normed spaces. We study $[V,\bar{\lambda}]$- and $[C,1]$-summabilities for double sequences. In addition, we obtain the relation between these concepts and $\bar{\lambda}$-statistically convergence.

Key words: *$\bar{\lambda}$-convergence, double sequence, summabilities*

Introduction

Theory of statistical convergence was firstly originated by Fast [1]. After Fridy [2] and Salat [3], statistical convergence became a notable topic in summability theory.

Fuzzy set theory has become an important working area for 40 years. It has been used in many engineering applications, control of chaos, non-linear operator, and population changes. It affected many mathematicians to investigate new kinds of sequence spaces and to study this type convergence. Fuzzy norm idea was firstly considered by Katsaras [4] in the fuzzy topological vector spaces. Also, Alimohammady and Roohi [5] introduced compactness in fuzzy minimal spaces. Felbin [6] was inspired by Kaleva and Seikkala [7], and then he introduced fuzzy norm of the linear space. Topological characterizations of fuzzy normed linear spaces were found in [8, 9]. Other studies on the same spaces can also be found in [10, 11].

The convergence of a sequence using fuzzy numbers was given by Matloka [12]. Nanda [13] studied the seqeuences with fuzzy numbers. He also formed complete metric space using set of all convergent sequences with fuzzy numbers. Mursaleen and Edely [14] introduced statistical convergence via double sequences and identified some results about statistical convergence. Cakan and Altay [15], Altay and Basar [16], and Tripaty [17] studied double sequences in summability theory. Statistical convergence using fuzzy numbers was given by Nuray and Savas [18]. Savas and Mursaleen [19] studied statistical convergent double sequences via fuzzy numbers. Sencimen and Pehlivan [20] introduced statistically convergent (resp. statistically Cauchy) sequence in the fuzzy normed linear space. Considering [20], Mohiuddine *et al.* [21] studied double sequences in the same spaces.

Mursaleen [22] introduced the idea of λ-statistical convergence. Using fuzzy numbers, λ-statistical convergence was presented Savas [23]. The λ-statistical convergence by using double sequences was obtained by Savas [24], and Savas and Patterson [25, 26]. Turkmen and Cinar [27] considered λ-statistical convergence within fuzzy normed linear space.

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Main results

We define double $\bar{\lambda}$-statistically convergence sequence, double $\bar{\lambda}$-statistically Cauchy sequence, $[V,\bar{\lambda}]$- and $[C,1]$-summabilities for double sequences in the fuzzy normed spaces. Also, we give notable properties and relationships between these concepts.

Throughout the paper, we consider $(X,\|\cdot\|)$ be and *FNS* and $\bar{\lambda}_{r,s}=\lambda_r\mu_s$ be the collection of such sequences $\bar{\lambda}$ will be represented by Δ_2.

Let $\lambda=(\lambda_r)$ and $\mu=(\mu_s)$ be non-decreasing sequences of $\mathbb{R}^+$, each converges to ∞ and such that $\lambda_{r+1}\le\lambda_r+1$, $\lambda_1=1$, $\mu_{s+1}\le\mu_s+1$, $\mu_1=1$. Let $I_r=[r-\lambda_r+1,\,r]$, $I_s=[s-\mu_s+1,\,s]$ and $I_{r,s}=I_r\times I_s$.

For any set $X\subset\mathbb{N}\times\mathbb{N}$, following number:

$$\delta_{\bar{\lambda}}(X)=P-\lim_{r,s\to\infty}\frac{1}{\bar{\lambda}_{r,s}}\left|(k,l)\in I_{r,s}:(k,l)\in X\right|$$

is called $\bar{\lambda}$-density of *X,* where the limit exists and then $\bar{\lambda}_{r,s}=\lambda_r\mu_s$.

Next, we give double $\bar{\lambda}$-statistically convergence in fuzzy normed space.

Definition 1. $x=(x_{k,l})$ in *X* is called double $\bar{\lambda}$-statistically convergent to $L\in X$ with regards to fuzzy norm on *X* or $FS_{\bar{\lambda}}$-convergent if for each $\varepsilon>0$:

$$\lim_{r,s\to\infty}\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l)\in I_{r,s}:D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|=0$$

It is demonstrated by

$$x_{k,l}\overset{FS_{\bar{\lambda}}}{\to}L\quad\text{or}\quad x_{k,l}\to L\left(FS_{\bar{\lambda}}\right)\quad\text{or}\quad FS_{\bar{\lambda}}-\lim_{k,l\to\infty}x_{k,l}=L$$

where $I_r=[r-\lambda_r+1,r]$, $I_s=[s-\mu_s+1,s]$ and $I_{r,s}=I_r\times I_s$.

For each $\varepsilon>0$:

$$K(\varepsilon)=\left\{(k,l)\in I_{r,s}:\left\|x_{k,l}-L\right\|_0^+\ge\varepsilon\right\}$$

has natural zero density. That is, $\|x_{k,l}-L\|_0^+<\varepsilon$ for almost all *k, l*. It is denoted:

$$FS_{\bar{\lambda}}-\lim_{k,l\to\infty}x_{k,l}=L$$

In *X,* all $\bar{\lambda}$-statistically convergent sequences is described by $FS_{\bar{\lambda}}(X)$, and also defined:

$$FS_{\bar{\lambda}}(X)=\left\{x=(x_{k,l}):\ \text{for some } L,\ \ FS_{\bar{\lambda}}-\lim_{k,l\to\infty}x_{k,l}=L\right\}$$

where $L\in X$ is $FS_{\bar{\lambda}}$-limit of a $(x_{k,l})$.

In terms of neighborhoods, we have

$$x_{k,l}\overset{FS_{\bar{\lambda}}}{\to}L$$

if for each $\varepsilon>0$:

$$\delta_{\bar{\lambda}}\left(\left\{(k,l)\in I_{r,s}:x_{k,l}\notin\mathcal{N}_L(\varepsilon,0)\right\}\right)=0$$

that is, for each $\varepsilon>0$, $x_{k,l}\notin\mathcal{N}_L(\varepsilon,0)$ for almost all (k,l).

A useful result of the definition is:

$$x_{k,l} \to L(FS_{\bar{\lambda}}) \quad \text{if} \quad FS_{\bar{\lambda}} - \lim_{k,l\to\infty} \| x_{k,l} - L \|_0^+ = 0$$

Note that $FS_{\bar{\lambda}} - \lim_{k,l\to\infty} \| x_{k,l} - L \|_0^+ = 0$ implies that:

$$FS_{\bar{\lambda}} - \lim_{k,l\to\infty} \left\| x_{k,l} - L \right\|_\alpha^- = FS_{\bar{\lambda}} - \lim_{k,l\to\infty} \left\| x_{k,l} - L \right\|_\alpha^+ = 0$$

for each $\alpha \in [0,1]$ since:

$$0 \le \left\| x_{k,l} - L \right\|_\alpha^- \le \left\| x_{k,l} - L \right\|_\alpha^+ \le \left\| x_{k,l} - L \right\|_0^+$$

holds for every $(k,l) \in I_{r,s}$ and for each $\alpha \in [0,1]$.

For all *r,s,* if $\bar{\lambda}_{r,s} = rs$, set of $\bar{\lambda}$-statistically convergent obtained by double sequences transforms to set of statistically convergent via double sequences in *X*.

Definition 2. $x = (x_{k,l})$ in *X* is called strongly double $\bar{\lambda}$-summable with regards to fuzzy norm on *X* if there is a $L \in X$ such that:

$$\lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \sum_{k,l\in I_{r,s}} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) = 0$$

It is denoted:

$$x_{k,l} \overset{[V,\bar{\lambda}]_{FN}}{\to} L \quad \text{or} \quad x_{k,l} \to L\left(\left[V, \bar{\lambda} \right]_{FN} \right) \quad \text{or} \quad \left[V, \bar{\lambda} \right]_{FN} - \lim_{k,l\to\infty} x_{k,l} = L$$

If $\bar{\lambda}_{r,s} = rs$, strongly double $\bar{\lambda}$-summable tranforms to $[C,1]_{FN}$, the space of strongly Cesaro summable with double sequences in fuzzy normed space. It is defined:

$$\lim_{r,s\to\infty} \sum_{k,l=1,1}^{r,s} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) = 0$$

So, we have written:

$$\left[V, \bar{\lambda} \right]_{FN} (X) = \left\{ x = (x_{k,l}) \ : \ \lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \sum_{k,l\in I_{r,s}} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) = 0, \text{ for some } L \right\}$$

$$[C,1]_{FN} (X) = \left\{ x = (x_{k,l}) \ : \ \lim_{r,s\to\infty} \frac{1}{rs} \sum_{k,l\in 1,1}^{r,s} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) = 0, \text{ for some } L \right\}$$

Definition 3. $x = (x_{k,l})$ in *X* is called $FS_{\bar{\lambda}}$-convergent to $L \in X$ with regards to fuzzy norm on *X* if for each $\varepsilon > 0$ and $t \in (0,1)$,

$$\delta_{\bar{\lambda}} = \left(\left\{ (k,l) \in I_{r,s} : D\left(\left\| x_{k,l} - L \right\|, \tilde{0} \right) \le 1 - t \right\} \right) = 0$$

or equivalently:

$$\delta_{\bar{\lambda}} = \left(\left\{ (k,l) \in I_{r,s} : D\left(\left\| x_{k,l} - L \right\|, \tilde{0} \right) > 1 - t \right\} \right) = 1$$

Theorem 1. If $FS_{\bar{\lambda}} - \lim x = L$ exists, it is unique.

Proof. Let L_1, L_2 $(L_1 \ne L_2)$ be in *X* such that:

$$FS_{\bar{\lambda}} - \lim x = L_1; \quad FS_{\bar{\lambda}} - \lim x = L_2$$

If $L_1 \neq L_2$, then $L_1 - L_2 \neq 0$. Therefore, $\| L_1 - L_2 \|_0^+ = 2\varepsilon > 0$, and we take a norm as $\| . \|_0^+$. Since $FS_{\bar{\lambda}} - \lim x = L_1$ and $FS_{\bar{\lambda}} - \lim x = L_2$ it follows that:

$$\lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \left\| x_{k,l} - L_1 \right\|_0^+ \geq \varepsilon \right\} \right| = 0$$

and

$$\lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \left\| x_{k,l} - L_2 \right\|_0^+ \geq \varepsilon \right\} \right| = 0$$

There are $(k,l) \in I_{r,s}$ such that:

$$\left\| x_{k,l} - L_1 \right\|_0^+ < \varepsilon \quad \text{and} \quad \left\| x_{k,l} - L_2 \right\|_0^+ < \varepsilon$$

Further, for these *k, l* we have:

$$\left\| L_1 - L_2 \right\|_0^+ \leq \left\| x_{k,l} - L_1 \right\|_0^+ + \left\| x_{k,l} - L_2 \right\|_0^+ < 2\varepsilon$$

which is a contradiction. Therefore, the limit unique.

Theorem 2. Let $(X, \| . \|)$ be an *FNS,* $\bar{\lambda} \in \Delta_2$. Then:

(i) If $x_{k,l} \to L([V,\bar{\lambda}]_{FN})$, then $x_{k,l} \to L(FS_{\bar{\lambda}})$ with regards to fuzzy norm on *X*.

(ii) $[V,\bar{\lambda}]_{FN}(X)$ is proper subset of $FS_{\bar{\lambda}}(X)$.

Proof. (i) Let $\varepsilon > 0$ and $x_{k,l} \to L([V,\bar{\lambda}]_{FN})$, we can write:

$$\sum_{k,l\in I_{r,s}} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \sum_{\substack{k,l\in I_{r,s} \\ (D\|x_{k,l}-L\|,\tilde{0})\geq\varepsilon}} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \varepsilon \left| \left\{ (k,l) \in I_{r,s} \ : \ \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \varepsilon \right\} \right|$$

and so:

$$\frac{1}{\varepsilon \bar{\lambda}_{r,s}} \sum_{k,l\in I_{r,s}} \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ \left(D\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \varepsilon \right\} \right|$$

This implies that if $x_{k,l} \to L([V,\bar{\lambda}]_{FN})$, then $x_{k,l} \to L(FS_{\bar{\lambda}})$ in fuzzy normed space. This completes the proof.

(ii) To show the inclusion $[V,\bar{\lambda}]_{FN}(X) \subset FS_{\bar{\lambda}}(X)$ is a proper, we define $x = (x_{k,l})$:

$$x_{k,l} = \begin{cases} (k,l), & \text{if} \quad k_{r-1} \leq k < k_{r-1} + \sqrt{\lambda_r}, \ l_{s-1} \leq l < l_{s-1} + \sqrt{\eta_s}, \quad (r,s = 1,2,...) \\ (0,0), & \text{otherwise} \end{cases}$$

We see that *x* is not bounded. For every $\varepsilon > 0$, $x \in X$, we obtain:

$$\frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D\left(\left\| x_{k,l} - L \right\|, \tilde{0} \right) \geq \varepsilon \right\} \right| \leq \frac{\left[\!\left[\sqrt{\lambda_r} \right]\!\right] \left[\!\left[\sqrt{\mu_s} \right]\!\right]}{\bar{\lambda}_{r,s}} \to 0, \quad (r,s \to \infty)$$

That is,

$$x_{k,l} \overset{FS_{\bar{\lambda}}}{\to} 0$$

On the other side:

$$\frac{1}{\bar{\lambda}_{r,s}}\sum_{k,l\in I_{r,s}}\left(D\left\|x_{k,l}-0\right\|,\tilde{0}\right)=\frac{1}{\bar{\lambda}_{r,s}}\sum_{k,l\in I_{r,s}}\left\|x_{k,l}\right\|=$$

$$=\frac{1}{\bar{\lambda}_{r,s}}\frac{\left[\sqrt{\lambda_r}\right]\left[\left(\sqrt{\lambda_r}+1\right)\right]\left[\sqrt{\mu_s}\right]\left[\left(\sqrt{\mu_s}+1\right)\right]}{4}\to\frac{1}{4}\neq 0$$

Hence, $x_{k,l}\nrightarrow 0([V,\bar{\lambda}]_{FN})$.

Theorem 3. Let a bounded $(x_{k,l})$ is double $\bar{\lambda}$-statistically convergent to L. Hence, it is strongly double $\bar{\lambda}$-summable to L. Therefore, $(x_{k,l})$ is double Cesaro summable to L with regards to fuzzy norm on X.

Proof. Assume that $(x_{k,l})$ is bounded and $x_{k,l}\to L(FS_{\bar{\lambda}})$. Since $(x_{k,l})$ is bounded, we get $D(\| x_{k,l}-L\|,\tilde{0})\leq M$ for all k, l. For $\varepsilon>0$ and for large r, s we get:

$$\frac{1}{\bar{\lambda}_{r,s}}\sum_{k,l\in I_{r,s}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)=$$

$$=\frac{1}{\bar{\lambda}_{r,s}}\sum_{\substack{k,l\in I_{r,s}\\(D\|x_{k,l}-L\|,\tilde{0})\geq\frac{\varepsilon}{2}}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)+\frac{1}{\bar{\lambda}_{r,s}}\sum_{\substack{k,l\in I_{r,s}\\(D\|x_{k,l}-L\|,\tilde{0})<\frac{\varepsilon}{2}}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\leq$$

$$\leq\frac{M}{\bar{\lambda}_{r,s}}\left|\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}$$

This implies that $x_{k,l}\to L([V,\bar{\lambda}]_{FN})$.

Further, we have:

$$\frac{1}{rs}\sum_{k,l\in 1,1}^{r,s}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)=\frac{1}{rs}\sum_{k,l\in 1,1}^{r-\lambda_r,s-\mu_s}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)+\frac{1}{rs}\sum_{k,l\in I_{r,s}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\leq$$

$$\leq\frac{1}{\bar{\lambda}_{r,s}}\sum_{k,l\in 1,1}^{r-\lambda_r,s-\mu_s}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)+\frac{1}{\bar{\lambda}_{r,s}}\sum_{k,l\in I_{r,s}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\leq$$

$$\leq\frac{2}{\bar{\lambda}_{r,s}}\sum_{k,l\in I_{r,s}}\left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)$$

Hence, $(x_{k,l})$ is double Cesaro summable to L since $(x_{k,l})$ strongly double $\bar{\lambda}$-summable to L in fuzzy normed space.

Theorem 4. Let $(x_{k,l})$ be double statistically convergent to L in fuzzy normed space. Then, it is double $\bar{\lambda}$-statistically convergent to L with regards to the fuzzy norm on X iff:

$$P-\lim_{r,s\to\infty}\inf\frac{\bar{\lambda}_{r,s}}{rs}>0 \tag{1}$$

Proof. For $\varepsilon>0$, we obtain:

$$\left\{(k,l),\ k\leq r,l\leq s\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\geq\varepsilon\right\}\supseteq\left\{(k,l)\in I_{r,s}\ :\ D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\geq\varepsilon\right\}$$

Therefore,

$$\frac{1}{rs}\left|\left\{(k,l),\ k\le r, l\le s\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|\ge\frac{1}{rs}\left|\left\{(k,l)\in I_{r,s}\ :\ D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|\ge$$

$$\ge\frac{\bar{\lambda}_{r,s}}{rs}\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|$$

Using limit as $r,s\to\infty$, we get:

$$x_{k,l}\overset{FS_{\bar{\lambda}}}{\to}L$$

That is,

$$x_{k,l}\overset{st_2(FN)}{\to}L\Rightarrow x_{k,l}\overset{FS_{\bar{\lambda}}}{\to}L$$

Conversely, assume that $x\in st_2(FN)(X)$ and since $\bar{\lambda}_{r,s}=\lambda_r\mu_s$ either $P-\lim_{r\to\infty}(\lambda_r/r)=$ $=0$ or $P-\lim_{s\to\infty}(\lambda_s/s)=0$ or both of them are zero. Hence, choosing subsequences $[r(p)]_{p=1}^{\infty}$ and $[s(q)]_{q=1}^{\infty}$ such that $[\lambda_{r_{(p)}}/r(p)]<1/p$ and $[\lambda_{s_{(q)}}/s(q)]<1/q$, we can describe a double sequence $x=(x_{k,l})$ by:

$$(x_{kl})=\begin{cases}1, & \text{if } k\in I_{r_{(p)}}, l\in I_{s_{(q)}}, (p,q=1,2,....),\\ 0, & \text{otherwise}\end{cases}$$

Therefore, $(x_{k,l})\in[C,1]_{FN}$ and $(x_{k,l})\in st_2(FN)(X)$. On the other hand, $x\notin[V,\bar{\lambda}]_{FN}(X)$. *Theorem 3* implies that $x\notin FS_{\bar{\lambda}}(X)$, a contradiction and hence of eq. (1) must hold.

Theorem 5. Let $(X,\|.\|)$ be an FNS and if $\bar{\lambda}\in\Delta_2$ such that $\lim_{r\to\infty}(\bar{\lambda}_{r,s}/rs)=1$. Then $FS_{\bar{\lambda}}(X)\subset st_2(FN)(X)$.

Proof. Since $\lim_{r\to\infty}(\bar{\lambda}_{r,s}/rs)=1$, we see following:

$$\frac{1}{rs}\left|\left\{(k,l),\ k\le r, l\le s\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|\le$$

$$\le\frac{1}{rs}\left|\left\{(k,l),\ k\le r-\lambda_r, l\le s-\mu_s\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|+$$

$$+\frac{1}{rs}\left|\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|\le$$

$$\le\frac{rs-\bar{\lambda}_{r,s}}{rs}+\frac{1}{rs}\left|\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|=$$

$$=\frac{rs-\bar{\lambda}_{r,s}}{rs}+\frac{\bar{\lambda}_{r,s}}{rs}\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\ge\varepsilon\right\}\right|$$

for $\varepsilon>0$. This implies that

$$x_{k,l}\overset{st_2(FN)}{\to}L,\quad\text{if}\quad x_{k,l}\overset{FS_{\bar{\lambda}}}{\to}L$$

Hence, $FS_{\bar{\lambda}}(X)\subset st_2(FN)(X)$.

Remark 1. We could not show whether the condition $\lim_{r\to\infty}(\bar{\lambda}_{r,s}/rs)=1$ in the previous theorem is necessary. We left it as an open problem.

Theorem 6. $FS_{\bar{\lambda}}-\lim_{k,l\to\infty}x_{k,l}=L$ if there is a subset:

$$K=\left\{(k_n,l_n):k_1<k_2<\ldots;l_1<l_2<\ldots\right\}\subseteq\mathbb{N}\times\mathbb{N}$$

such that $\delta_{\bar{\lambda}}(K)=1$ and $st_2(FN)-\lim_{r,s\to\infty} x_{k_r l_s}=L$.

Proof. Assume that $FS_{\bar{\lambda}}-\lim_{k,l\to\infty} x_{k,l}=L$. Hence, for any $\varepsilon>0$ and $s\in\mathbb{N}$, let:

$$K(s,\varepsilon)=\left\{(k,l)\in I_{r,s}: D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\le 1-\frac{1}{s}\right\}$$

$$M(s,\varepsilon)=\left\{(k,l)\in I_{r,s}: D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)> 1-\frac{1}{s}\right\}$$

Then $\delta_{\bar{\lambda}}[K(s,\varepsilon)]=0$ and

$$M(1,\varepsilon)\supset M(2,\varepsilon)\supset\ldots\supset M(i,\varepsilon)\supset M(i+1,\varepsilon)\supset\ldots \tag{2}$$

and

$$\delta_{\bar{\lambda}}\left[M(s,\varepsilon)\right]=1,\ s=1,2,\ldots$$

Now we have to show for $(k,l)\in M(s,\varepsilon)$, $x=(x_{k,l})$ is $st_2(FN)$-convergent to L. Let $x=(x_{k,l})$ is not $st_2(FN)$-convergent to L for some $(k,l)\in M(s,\varepsilon)$. Therefore, there is $t>0$ and are positive integers l_0,k_0 such that $D([x_{k,l}-L],\tilde{0})\le 1-t$ for all $l\ge l_0,k\ge k_0$. And let $(\|x_{k,l}-L\|,\tilde{0})>1-t$ for all $l\le l_0,k\le k_0$, hence:

$$\delta_{\bar{\lambda}}\left(\left\{(k,l)\in I_{r,s}:\left[D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\right]>1-t\right\}\right)=0$$

Since $t>1/s$, we obtain:

$$\delta_{\bar{\lambda}}[M(s,\varepsilon)]=0$$

which contradicts to eq. (2). Finally, $x=(x_{k,l})$ is double statistical convergent to L.

Conversely, suppose that there is a subset:

$$K=\left\{(k_n,l_n): k_1<k_2<\ldots;l_1<l_2<\ldots\right\}\subseteq\mathbb{N}\times\mathbb{N}$$

such that $\delta_{\bar{\lambda}}(K)=1$ and $st_2(FN)-\lim_{r,s\to\infty} x_{k_r l_s}=L$. There exists $N\in\mathbb{N}$ for every $t\in(0,1)$ and $\varepsilon>0$,

$$\left[D\left(\left\|x_{k,l}-L\right\|,\tilde{0}\right)\right]>1-t,\quad \text{for all}\quad k,l\ge\mathbb{N}$$

Next,

$$M(t,\varepsilon)=\left\{(k,l)\in I_{r,s}\ :\ \left(D\left\|x_{k,l}-L\right\|,\tilde{0}\right)\le 1-t\right\}\subseteq$$
$$\subseteq\mathbb{N}\times\mathbb{N}-\left\{\left(k_{N+1},L_{N+1}\right),\left(k_{N+2},L_{N+2}\right),\ldots\right\}$$

Therefore, $\delta_{\bar{\lambda}}[M(s,\varepsilon)]\le 1-1=0$. Hence, we have $FS_{\bar{\lambda}}-\lim_{k,l\to\infty} x_{k,l}=L$.

Definition 4. Let $(X,\|.\|)$ be an *FNS,* $\bar{\lambda}\in\Delta_2$. The double sequence $x=(x_{k,l})$ is called $\bar{\lambda}$-statistically Cauchy sequence with regards to fuzzy norm on X if there is a double subsequence $x=\{x_{\bar{k}_r,\bar{l}_s}\}$ of x such that $(\bar{k}_r,\bar{l}_s)\in I_{r,s}$ for each (r,s), $FS_{\bar{\lambda}}-\lim_{r,s\to\infty} x_{\bar{k}_r,\bar{l}_s}=L$ and for every $\varepsilon>0$:

$$\lim_{r,s\to\infty}\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\left[D\left(\left\|x_{k,l}-x_{\bar{k}_r,\bar{l}_s}\right\|,\tilde{0}\right)\right]>\varepsilon\right\}\right|=0$$

Theorem 7. The double sequence $x=(x_{k,l})$ in X is $\bar{\lambda}$-statistically convergent with regards to fuzzy norm on X if $x=(x_{k,l})$ is $\bar{\lambda}$-statistically Cauchy sequence.

Proof. Let $FS_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = L$ and

$$K^{t,v} = \left\{(k,l) \in I_{r,s} : \left[D\left(\left\|x_{k,l} - L\right\|, \tilde{0}\right)\right] < \frac{1}{tv}\right\}$$

We obtain the following:

$$K^{t+1,v+1} \subseteq K^{t,v} \quad \text{and} \quad \frac{\left|K^{t,v} \cap I_{r,s}\right|}{\bar{\lambda}_{r,s}} \to 1, \quad \text{as} \quad r,s \to \infty$$

This implies that there is k_1 and l_1 such that $r \geq k_1$ and $s \geq l_1$ and $|K^{1,1} \cap I_{r,s}| / \bar{\lambda}_{r,s} > 0$, that is, $K^{1,1} \cap I_{r,s} \neq \varnothing$. We next choose $k_2 \geq k_1$ and $l_2 \geq l_1$ such that $r \geq k_2$ and $s \geq l_2$ implies that $K^{2,2} \cap I_{r,s} \neq \varnothing$. Thus, for each pair (r,s) such that $k_1 \leq r < k_2$ and $l_1 \leq s < l_2$. We select $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ such that $(\bar{k}_r, \bar{l}_s) \in K^{r,s} \cap I_{r,s}$ that is:

$$\left[D\left(\left\|x_{\bar{k}_r,\bar{l}_s} - L\right\|, \tilde{0}\right)\right] < 1$$

In general, we choose $k_{t+1} \geq k_t$ and $l_{v+1} \geq l_v$ such that $r > k_{t+1}$ and $s > l_{v+1}$. This implies $K^{t+1,v+1} \cap I_{r,s} \neq \varnothing$. Thus, for all (r,s) such that for $k_t \leq r < k_{t+1}$ and $l_v \leq s < l_{v+1}$ choose $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$, that is:

$$\left[D\left(\left\|x_{k,l} - L\right\|, \tilde{0}\right)\right] < \frac{1}{tv}$$

Thus, $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ for each pair (r,s) and:

$$\left[D\left(\left\|x_{\bar{k}_r,\bar{l}_s} - L\right\|, \tilde{0}\right)\right] < \frac{1}{tv}$$

implies $FS_{\bar{\lambda}} - \lim_{r,s\to\infty} x_{\bar{k}_r,\bar{l}_s} = L$. Also, for each $\varepsilon > 0$:

$$\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{k,l} - x_{\bar{k}_r,\bar{l}_s}\right\|, \tilde{0}\right) \geq \varepsilon\right\}\right| \leq \frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{k,l} - L\right\|, \tilde{0}\right) \geq \tfrac{\varepsilon}{2}\right\}\right| +$$
$$+\frac{1}{\bar{\lambda}_{r,s}}\left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{\bar{k}_r,\bar{l}_s} - L\right\|, \tilde{0}\right) \geq \tfrac{\varepsilon}{2}\right\}\right|$$

Since $FS_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = L$ and $FS_{\bar{\lambda}} - \lim_{r,s\to\infty} x_{\bar{k}_r,\bar{l}_s} = L$, it follows that x is $\bar{\lambda}$-statistically Cauchy sequence.

Next, we assume x is an $\bar{\lambda}$-statistically Cauchy sequence. Then:

$$\left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{k,l} - L\right\|, \tilde{0}\right) \geq \frac{\varepsilon}{2}\right\}\right| \leq \left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{k,l} - x_{\bar{k}_r,\bar{l}_s}\right\|, \tilde{0}\right) \geq \frac{\varepsilon}{2}\right\}\right| +$$
$$+\left|\left\{(k,l) \in I_{r,s} \ : \ D\left(\left\|x_{\bar{k}_r,\bar{l}_s} - L\right\|, \tilde{0}\right) \geq \frac{\varepsilon}{2}\right\}\right|$$

Therefore, $FS_{\bar{\lambda}} - \lim_{k,l\to\infty} x_{k,l} = L$. Thus, the theorem is proved.

Theorem 8. $FS_{\bar{\lambda}}(X) \cap l_2^{\infty}(X)$ is a closed subset of $l_2^{\infty}(X)$, if X is a fuzzy normed Banach space.

Proof. Let $(x^n)_{n\in\mathrm{N}} = (x_{k,l}^n)_{k,l\in\mathrm{N}}$ be a convergent sequence in $FS_{\bar{\lambda}}(X) \cap l_2^{\infty}(X)$ converging to $x \in l_2^{\infty}(X)$. We have to prove that $x \in FS_{\bar{\lambda}}(X) \cap l_2^{\infty}(X)$. And let:

$$\left(x_{k,l}^{n}\right) \xrightarrow{FS_{\bar{\lambda}}} L_n ,$$

for all $n \in \mathbb{N}$. Taking a positive decreasing convergent sequence $(\varepsilon_n)_{n\in\mathrm{N}}$, where $\varepsilon_n = \varepsilon/2^n$, for a given $\varepsilon > 0$, we select $(\varepsilon_n)_{n\in\mathrm{N}}$ which converges to zero. Choosing $n \in \mathbb{Z}^+$, such that $\| x - x^n \|_\infty < \varepsilon_n/4$ for all $n \geq r, s$, then we get:

$$\lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \right\} \right| = 0$$

and

$$\lim_{r,s\to\infty} \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D\left(\left\| x_{k,l}^{n+1} - L_{n+1} \right\|, \tilde{0} \right) \geq \frac{\varepsilon_{n+1}}{4} \right\} \right| = 0$$

Since

$$\frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \vee D\left(\left\| x_{k,l}^{n+1} - L_{n+1} \right\|, \tilde{0} \right) \geq \frac{\varepsilon_{n+1}}{4} \right\} \right| < 1$$

for all $k, l \in \mathbb{N}$,

$$\left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \right\} \cap \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n+1} - L_{n+1}, \tilde{0} \right\| \right) \geq \frac{\varepsilon_{n+1}}{4} \right\}$$

is infinite. Hence, there must exist $(k,l) \in I_{r,s}$ for which we have, simultaneously:

$$D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) < \frac{\varepsilon_n}{4} \quad \text{and} \quad D\left(\left\| x_{k,l}^{n} - L_{n+1} \right\|, \tilde{0} \right) < \frac{\varepsilon_{n+1}}{4}$$

It follows that:

$$D\left(\left\| L_n - L_{n+1} \right\|, \tilde{0} \right) \leq D\left(\left\| L_n - x_{k,l}^{n} \right\|, \tilde{0} \right) + D\left(\left\| x_{k,l}^{n} - x_{k,l}^{n+1} \right\|, \tilde{0} \right) + D\left(\left\| x_{k,l}^{n+1} - L_{n+1} \right\|, \tilde{0} \right) \leq$$

$$\leq D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) + D\left(\left\| x_{k,l}^{n+1} - L_{n+1} \right\|, \tilde{0} \right) + D\left(\left\| x - x^n \right\|_\infty, \tilde{0} \right) + D\left(\left\| x - x^{n+1} \right\|_\infty, \tilde{0} \right) \leq$$

$$\leq \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \varepsilon_n$$

It gives (L_n) is a Cauchy sequence. We can write $L_n \to L \in X$ as $n \to \infty$, since X is a fuzzy normed Banach space. We prove that $x_{k,l} \to L(FS_{\bar{\lambda}})$. For any $\varepsilon > 0$, taking $n \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon/4$:

$$D\left(\left\| x_{k,l} - x_{k,l\infty}^{n} \right\|, \tilde{0} \right) < \frac{\varepsilon}{4} \quad \text{and} \quad D\left(\left\| L_n - L \right\|, \tilde{0} \right) < \frac{\varepsilon}{4}$$

Hence, we have:

$$\frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l} - L \right\|_\infty, \tilde{0} \right) \geq \varepsilon \right\} \right| \leq \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) + \right.\right.$$

$$\left.\left. + D\left(\left\| x_{k,l} - x_{k,l}^{n} \right\|_\infty, \tilde{0} \right) + D\left(\left\| L_n - L \right\|, \tilde{0} \right) \geq \varepsilon \right\} \right| \leq$$

$$\leq \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon \right\} \right| \leq$$

$$\leq \frac{1}{\bar{\lambda}_{r,s}} \left| \left\{ (k,l) \in I_{r,s} \ : \ D\left(\left\| x_{k,l}^{n} - L_n \right\|, \tilde{0} \right) \geq \frac{\varepsilon}{2} \right\} \right| \to 0 \ \text{ as } \ r,s \to \infty$$

This gives $x_{k,l} \to L(FS_{\bar{\lambda}})$.

Conclusion

In this paper, we give $\bar{\lambda}$-statistically convergence, $\bar{\lambda}$-statistically Cauchy sequence, strongly $\bar{\lambda}$-summable for double sequences in fuzzy normed spaces. In further studies, the $\bar{\lambda}$-ideal convergence by using double sequences can be defined and examined in fuzzy normed spaces.

Acknowledgment

We thank to editor and referees for their notable reading, suggestions and also remarks.

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Paper submitted: September 30, 2018
Paper revised: November 4, 2018
Paper accepted: November 22, 2018