HE'S HOMOTOPY PERTURBATION METHOD FOR SOLVING TIME FRACTIONAL SWIFT-HOHENBERG EQUATIONS

by

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This paper find the most effective method to solve the time-fractional Swift-Hohenberg equation with cubic-quintic non-linearity by combining the homotopy perturbation method and the fractional complex transform. The solution reveals some intermittent properties of thermal physics.

Key words: *time fractional Swift-Hohenberg equation, homotopy perturbation method, fractional complex transform*

Introduction

The Swift-Hohenberg (S-H) equation is:

$$\frac{\partial u}{\partial t} = \left\{ \mu - \left(1 + \frac{\partial^2}{\partial x^2} \right)^2 \right\} u - u^3 \tag{1}$$

was originally introduced to describe the temperature and fluid velocity dynamics of the thermal convection [1], where μ is a parameter. The S-H equation plays a key role in different branches of thermal physics, and it has been widely studied [2-6]. Recently, the S-H equation with fractional time derivative is studied by Khan *et al.* [7], Merdan [8], and Vishal *et al.* [9].

Due to its important applications in engineering and thermal physics, the authors are motivated to solve the following S-H equation with cubic-quintic non-linearity:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \mu u - \left(1 + \frac{\partial^2}{\partial x^2} \right)^2 u + N(u), \qquad 0 < \alpha \le 1 \tag{2}$$

$$u(x,0) = f(x)$$

where $\partial^\alpha/\partial t^\alpha$ is the He's fractional derivative [10], $N(u) = \lambda u^3 - \sigma u^5$, and λ, σ, and μ are parameters. Equation (2) describes a thermal convection with intermittent property, that means the thermal convection is discontinuous with time. To understand this intermittent property, we consider evaporation in a river, during the daytime, evaporation happens, while at nighttime it does not happen. Such discontinuous property can be best described by fractional calculus. This paper aims at studying the intermittent property of the S-H equation.

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We will use a very reliable and efficient technique namely homotopy perturbation method (HPM) [11-13] with the fractional complex transform [14] to find approximate solutions of eq. (2). Our results reveal the complete reliability and efficiency of the proposed algorithm.

Homotopy perturbation method

The principals of the HPM and its applicability for various kinds of differential equations are given in [15-26]. For convenience of the reader, we will present a review of the HPM.

We consider the non-linear differential equation:

$$L(u) + N(u) = f(r), \qquad r \in \Omega \tag{3}$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma \tag{4}$$

where L is a linear operator, N – a non-linear operator, B – a boundary operator, Γ – the boundary of the domain Ω, and $f(r)$ – a known analytic function.

The homotopy perturbation technique defines the homotopy:

$$v(r, p) : \Omega \times [0,1] \to R$$

which satisfies:

$$H(v, p) = (1 - p)\left[L(v) - L(u_0)\right] + p\left[L(v) + N(v) - f(r)\right] = 0 \tag{5}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{6}$$

where $r \in \Omega$ and $p \in [0, 1]$ is an impeding parameter, u_0 – an initial approximation which satisfies the boundary conditions. Obviously, from eqs. (5) and (6), we have:

$$H(v, 0) = L(v) - L(u_0) = 0 \tag{7}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0 \tag{8}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from u_0 to $u(r)$. In topology, this called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic.

The basic assumption is that the solution of eqs. (3) and (4) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{9}$$

The approximate solution of eq. (3), therefore, can be obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{10}$$

The convergence of the series (10) has been proved in [24].

The solutions of eq. (2)

By the fractional complex transform [13, 14]:

$$s=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$$

Equation (2) is converted to a integer order differential equation:

$$\frac{\partial u}{\partial s}=\mu u-\left(1+\frac{\partial^2}{\partial x^2}\right)^2 u+N(u) \tag{11}$$

$$u(x,0)=f(x)$$

Equaton (11) can be re-writen:

$$u=f(x)+J\left[L(u)+N(u)\right] \tag{12}$$

where J is the integral operator with respect to s and:

$$L=\mu-\left(1+\frac{\partial^2}{\partial x^2}\right)^2$$

$$N(u)=\lambda u^3-\sigma u^5$$

For eq. (12), we can establish the following homotopy:

$$u=f(x)+pJ\left[L(u)+N(u)\right] \tag{13}$$

where p is the homotopy parameter.

Now, let us apply the HPM to eq. (13), according to which u can be expanded into power series in p:

$$u(x,s)=\sum_{n=0}^{\infty}p^n u_n(x,s) \tag{14}$$

and the non-linear term can be decomposed:

$$N(u)=\sum_{n=0}^{\infty}p^n H_n(u) \tag{15}$$

where $p\in[0, 1]$ is an embedding parameter and $H_n(u)$ is the He's polynomials [27] defined:

$$H_n(u_0,u_1,\cdots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0}, \qquad n=0,1,2,\ldots.$$

Here the first few terms of He's polynomials are given:

$$H_0(u_0)=u_0^3-u_0^5$$

$$H_1(u_0,u_1)=3u_0^2u_1-5u_0^4u_1$$

$$H_2(u_0,u_1,u_2)=3u_0^2u_2-5u_0^4+3u_0u_1^2-10u_0^3u_1^2$$

Substituting eqs. (14) and (15) into eq. (13), we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,s) = f(x) + pJ\left\{L\left[\sum_{n=0}^{\infty} p^n u_n(x,s)\right]\sum_{n=0}^{\infty} p^n H_n\right\}$$

Equating the terms with identical powers in p, we have:

$$u_0(x,s) = f(x)$$

$$u_1(x,s) = J[L(u_0) + H_0(u_0)]$$

$$u_2(x,s) = J[L(u_1) + H_1(u_0, u_1)]$$

and so on.

Applying backward substitution to the computed components $s = t^a/\Gamma(1+\alpha)$, we get $u_i(x, t)$, (i = 0, 1, 2, …) and finally the solution of the problem is obtained by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

For example, if $f(x) = \sin x$, and $\lambda = \sigma = 1$, then:

$$u_0(x,t) = \sin x$$

$$u_1(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}(\mu + \sin^2 x - \sin^4)\sin x$$

$$u_2(x,t) = \frac{t^{2\alpha}}{\Gamma^2(1+\alpha)}(\mu^2 \sin x + 4\mu \sin^3 x - 6\mu \sin^5 x -$$

$$-16\sin^3 x + 59\sin^5 x - 6\sin^7 x + 5\sin^9 x +$$

$$+ 48\sin x\cos^2 x + 120\sin x\cos^4 x - 400\sin^3 x\cos^2 x)$$

Preceding in this manner the rest of the components u_n, $n \geq 3$, can be completely obtained, and the series solutions are thus entirely determined.

Conclusion

We have solved the time fractional S-H equation with cubic-quintic non-linearity. The proposed algorithm here is a combination of the HPM, the fractional complex transform and He's polynomials. The intermittent property of the S-H equation depends upon the fractional order, when $\alpha = 0$, it becomes a stead solution, while when $\alpha = 1$ it turns out to the classic partner. The value of the fractional order implies intermittent period. We think that the proposed technology in this paper has great potential and can be applied to other strong non-linear differential equations.

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