EFFECTS OF CARBON NANOTUBES ON MAGNETOHYDRODYNAMIC FLOW OF METHANOL BASED NANOFLUIDS *VIA* ATANGANA-BALEANU AND CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES

by

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*This paper dedicatedly reports the heat transfer analysis of single and multi-walls carbon nanotubes for electrically conducting flow of Casson fluid. Both types of carbon nanotubes are suspended in methanol that is considered as a conventional base fluid. The governing PDE of nanofluids have been modeled by employing newly defined fractional approaches (derivatives) namely Atangana-Baleanu and Caputo-Fabrizio fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established via both approaches i. e. Atangana-Baleanu and Caputo-Fabrizio fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}^{y}_{\varepsilon,\delta}(T)$ and generalized **M**-function $\mathbf{M}^{p}_{q}(F)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases (Atangana-Baleanu and Caputo-Fabrizio fractional derivatives) are depicted for graphical illustrations. The comparison of three types of fluids: pure methanol, methanol with single walls carbon nanotubes, and methanol with multi-walls carbon nanotubes is portrayed via Atangana-Baleanu and Caputo-Fabrizio fractional derivatives. Finally, the results indicate that, pure methanol moves quicker in comparison with methanol-single-walls carbon nanotubes via Caputo-Fabrizio and methanol-multi-walls carbon nanotubes, while for larger time, these nanotubes move more rapidly in comparison with pure methanol and methanol-single-walls carbon nanotubes via Atangana-Baleanu.*

Key words: *Casson fluid, nanoparticles, MHD, fractional derivatives*

Introduction

The analysis of non-Newtonian liquids has attained significant consideration due to their immersion in extensive engineering and industrial applications. Such applications immerse plastics manufacturing, petroleum production, bioengineering, drawing of stretching sheet through quiescent fluid, food processing, polymeric liquids, aerodynamic extrusion of

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plastic films, annealing and thinning of copper wires, and several others. There is no denying fact that the Navier-Stokes expression does not characterize flows of non-Newtonian liquids. A single relation is not adequate to predict the characteristics and features of non-Newtonian liquids, even numerous non-Newtonian relations are available in vast literature [1-5]. Among the categories of non-Newtonian liquids, a Casson fluid is one of them. A Casson fluid has proved to be the best for the description of shear-thinning liquids having zero viscosity at an infinite rate of shear and infinite viscosity at zero shear rates [6, 7]. In continuation, the development of industrial manufacturing processes totally depends upon greater heat transfer rates. The normal techniques for heat transfer are not adequate to supply reasonable heat transfer rates for industrial and manufacturing needs. In order to develop the technique for enhancing heat transfer, several scientists, mathematicians, engineers, and numerical analysts are working in this field. Their main purpose is to enhance the performance of various liquids for instance, water, ethylene-glycol, oil, and few others. The abrasion, clogging, additional pressure loss, *etc.* are the aspects which are inadequate to enhance thermal conductivity. The idea of nanofluids was proposed by Choi [8], he verified through his experimental work that thermal performance of carrier-liquid can be enhanced *via* merging/submersion of (tiny) small size metallic or solid particles. The mechanism of nanofluids *via* nanoparticles in base fluid was investigated by Buongiorno [9]. In this mechanism, nanoparticles include size, inertia, magnus effect, Brownian motion, particle agglomeration, volume fraction, and thermophoresis. On the other hand, nanofluids are utilized to enhance heat transfer rate and thermal conductivity of base fluids [10-18]. For this purpose, different researchers utilize various types of nanoparticles with distinct sizes and shapes. Most the nanoparticles are made up of oxides, metals, carbon nanotubes (CNT), carbides *etc.* while engine oil, kerosene, ethylene glycol and water are considered as the base fluids. The CNT are mainly divided into two categories: single walls CNT (SWCNT) and multiple walls CNT (MWCNT) as in fig. 1 [67].

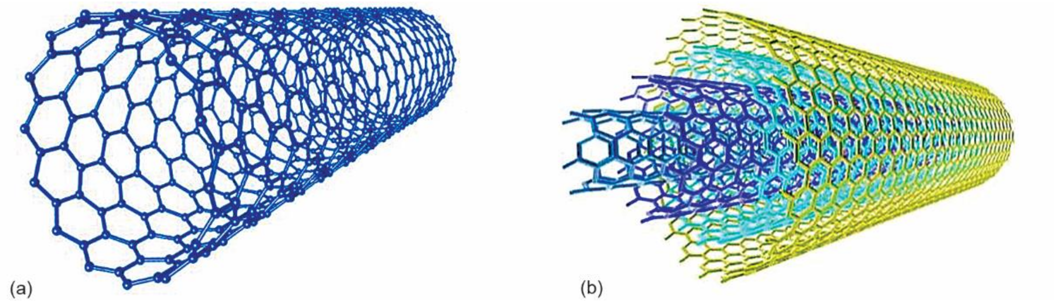

Figure 1. (a) SWCNT, (b) MWCNT

In brevity, CNT have gotten significant attention due to highest thermal conductivity in comparison with other nanostructure materials. The thermal conductivity of SWCNT and MWCNT is substantially higher than the thermal conductivity of metal oxide nanoparticles or metal nanoparticles. The SWCNT and MWCNT have remarkable thermal, optical, electrical, and mechanical properties due to cylindrical carbon molecules origin. Hence, they have been declared as best tools for special thermal properties having extra thermal conductivities, this is due to the facts enumerated: the diameter of SWCNT and MWCNT which ranges between 1 to 100 nm while length in micrometer, the SWCNT and MWCNT have two hundred times strength, the SWCNT and MWCNT have fifteen times thermal conductivity, the SWCNT and MWCNT have hundred times current capacity of copper, the SWCNT and MWCNT have five times elasticity of steel, the SWCNT and MWCNT have high aspect ratio which assist to form a network of conductive tubes, and the mechanical and electronic properties of SWCNT and MWCNT can be implemented for instance, leading-edge electronic fabrication, field-emission displays, nanosensors, nanocomposite materials. Even this, the Environmental Protection Agency has declared the SWCNT and MWCNT as the non-hazardous particles for the environment. However, in comparison with various other nanoparticles, several researchers are engaged to work on the SWCNT and MWCNT for the enhancement of thermal conductivity. Liu *et al.* [19] analyzed the single- and multi-WCNT for enhancing the thermal conductivity of engine oil and ethylene glycol. They inspected that ethylene glycol fluid without CNT have lower thermal conductivities in comparison with ethylene glycol fluid with CNT. Thermal conductivity of ethylene glycol fluid with CNT at volume fraction of 0.01 was increased by 12.4%, in contrast to this, thermal conductivity of engine oil with CNT at volume fraction of 0.02 was increased by 30%. Marquis and Chibnate [20] examined the improvement in the thermal conductivity of SWCNT and MWCNT with nanolubricants and nanofluids, in which they considered three distinct types of nanolubricants and nanofluids. Xie *et al.* [21] inspected nanofluids consisting of MWCNT for enhancing the thermal conductivities experimentally. They checked that increment in CNT into fluid generates enhancement in thermal conductivity. Khan *et al.* [22] analyzed the fluid problem with Navier slip boundary condition on heat transfer of CNT over flat plate. They investigated that engine oil and kerosene-based CNT have lower thermal conductivities and densities in comparison with water based CNT. Haq *et al.* [23] examined SWCNT and MWCNT for water-based flow. They perceived lower Nusselt number and skin friction for MWCNT in comparison with SWCNT. In the similar study, Haq *et al.* [24] investigated kerosene and water-based CNT fluid has lower heat transfer and skin friction in comparison with engine oil-based CNT fluid. Camilli *et al.* [25] presented experimental study of viscosity of CNT for water based nanofluids, in which they considered the impacts of volume fraction and temperature. Kamali and Binesh [26] explored numerical study for MWCNT with nanofluids using fixed wall heat flux condition in a straight tube. They observed that due to non-Newtonian behavior of CNT nanofluids, heat transfer coefficient is dominated by wall region. Of course, the list of studies on SWCNT and MWCNT, MHD, fractional derivatives, porous medium and nanoparticles with nanofluids [27-34] can be carried on, but we close it with some of the most interesting references published recently [35-46]. In the area of fluid mechanics and CFD, the fractional derivatives of non-integer orders are well-known. There are several types of fractional derivatives. These fractional derivatives have been recommended by prominent mathematicians as Riemann, Liouville, Sonin, Weyl, Letnikov, Erdelyi, Riesz, Kober, and many others. The fractional derivatives of non-integer orders are enthusiastically used in the fluid mechanics, CFD and natural sciences to evaluating the systems and processes with spatial and temporal non-locality (the non-locality in time is usually called memory). In this continuity, Saqib *et al.* [47] investigated free convection flow of Jeffrey fluid by employing Caputo-Fabrizio time-fractional derivatives and recovered the existing solutions in the open literature as well. Nadeem *et al.* [48] presented an interesting comparative analysis *via* Caputo and Fabrizio and Atangana and Baleanu fractional derivatives based on exponential and generalized Mittag-Leffler function as a kernel. They implemented the non-singular and non-local kernel on free convection flow of a generalized Casson fluid. Nadeem *et al.* [49] implemented modern fractional derivatives with the non-singular and non-local kernel to enhance the heat transfer rate of solar energy devices *via* nanoparticles. Syed *et al.* [50] analyzed Molybdenum Disulphide (MoS_2) nanoparticles of spherical shape and suspended in engine oil based generalized Brinkman-type nanofluid *via* newly introduced fractional derivatives known as Atangana-Baleanu derivative. Although studies of fluid phenomenon can be consistent yet we end here by citing recent attempts [50-54]. Motivating by previous research study, our aim is to report the heat transfer analysis of single- and multi-WCNT for electrically conducting flow of Casson fluid. Both types of CNT are suspended in methanol which is considered as a conventional base fluid. The governing PDE of nanofluids have been modeled by employing newly defined fractional derivatives namely Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established *via* both approaches *i. e.* AB and CF fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}^{y}_{\varepsilon,\delta}(T)$ and generalized $\mathbf{M}$-function $\mathbf{M}^{p}_{q}(F)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases, AB and CF fractional derivatives, are depicted for graphical illustrations. Finally, the comparison of three types of fluids: pure methanol, methanol with SWCNT, and methanol with MWCNT is portrayed *via* AB and CF fractional derivatives.

Governing equations

Let us consider an incompressible unsteady flow of Casson nanofluid containing CNT occupying a semi-finite space $y > 0$. The plate is assumed to be electrically conducting with a uniform magnetic field B of strength B_0, applied in a direction perpendicular to the plate. The magnetic Reynolds number is assumed to be small enough to neglect the effect of applied magnetic field. The CNT are suspended in methanol taken as base fluid. Initially, at time $t = 0$, both the fluid and the plate are at rest with constant temperature T_∞. At time $t = 0^+$, the plate is subjected to accelerated motion. At the same time, the plate temperature is raised to T_w which is thereafter maintained constant. Following [55], the equations governing the flow and heat transfer are given by:

$$\rho_\mathrm{nf}\frac{\partial F(y,t)}{\partial t}=\mu_\mathrm{nf}\left(\frac{1}{\beta}+1\right)\frac{\partial^2 F(y,t)}{\partial y^2}-\sigma_\mathrm{nf}B_0^2F(y,t)+\mathrm{g}(T-T_\infty)(\rho\beta_T)_\mathrm{nf} \tag{1}$$

$$(\rho c_p)_\mathrm{nf}\frac{\partial T(y,t)}{\partial t}=k_\mathrm{nf}\frac{\partial^2 T(y,t)}{\partial y^2} \tag{2}$$

where μ_nf is the dynamic viscosity, β – the Casson fluid parameter, ρ_nf – the density, σ_nf – the electrical conductivity, g – the gravitational acceleration, $(\rho\beta)_\mathrm{nf}$ – the volumetric thermal expansion coefficient, $(\rho c_p)_\mathrm{nf}$ – the heat capacitance, and k_nf – the thermal conductivity. The subscript nf is used for nanofluid, whereas f will be used for base fluid.

Initial and boundary conditions are:

$$\begin{aligned}&F(y,0)=0,\quad T(y,0)=T_\infty, \qquad y\ge 0\\&F(0,t)=At^p,\quad T(0,t)=T_w, \qquad t>0\\&F(\infty,t)=0,\quad T(\infty,t)=T_\infty, \qquad t>0\end{aligned} \tag{3}$$

were A is the arbitrary constant and its dimension depends on the value of p upon t. For constantly accelerated motion dimension of A will be [LT^{-2}], and for variably accelerated motion, dimension of A will be [LT^{-3}].

Using Xue [56] model for dynamic viscosity and thermal conductivity:

$$\frac{\mu_{\mathrm{nf}}}{\mu_{\mathrm{f}}}=\frac{1}{(1-\phi)^{2.5}}$$

$$\frac{k_{\mathrm{nf}}}{k_{\mathrm{f}}}=\frac{2\phi\left(\dfrac{k_{\mathrm{CNT}}}{k_{\mathrm{CNT}}-k_{\mathrm{f}}}\right)\ln\left(\dfrac{k_{\mathrm{f}}+k_{\mathrm{CNT}}}{2k_{\mathrm{f}}}\right)-\phi+1}{2\phi\left(\dfrac{k_{\mathrm{f}}}{k_{\mathrm{CNT}}-k_{\mathrm{f}}}\right)\ln\left(\dfrac{k_{\mathrm{f}}+k_{\mathrm{CNT}}}{2k_{\mathrm{f}}}\right)-\phi+1} \tag{4}$$

The ρ_{nf}, σ_{nf}, $(\rho\beta)_{\mathrm{nf}}$, and $(\rho c_p)_{\mathrm{nf}}$ are given by:

$$\rho_{\mathrm{nf}}=\phi\rho_{\mathrm{CNT}}+\rho_f(1-\phi)$$
$$\sigma_{\mathrm{nf}}=\left[1+\frac{\phi(3\sigma_{\mathrm{CNT}}-3)}{(2+\sigma_{\mathrm{CNT}})-\phi(\sigma_{\mathrm{CNT}}-1)}\right]\sigma_{\mathrm{f}},$$
$$(\rho\beta)_{\mathrm{nf}}=\phi(\rho\beta)_{\mathrm{CNT}}+(\rho\beta)_f(1-\phi),$$
$$(\rho c_p)_{\mathrm{nf}}=\phi(\rho c_p)_{\mathrm{CNT}}+(\rho c_p)_{\mathrm{f}}(1-\phi) \tag{5}$$

Introducing the following dimensionless variables:

$$t^*=\frac{tU_0^2}{\nu},\quad F^*=\frac{F}{U_0},\quad y^*=\frac{yU_0}{\nu},\quad T^*=\frac{T-T_\infty}{\Delta T};\quad \Delta T=T_w-T_\infty \tag{6}$$

into eqs. (1)-(4), one gets:

$$a_0\frac{\partial F(y,t)}{\partial t}=a_1\frac{\partial^2F(y,t)}{\partial y^2}-a_2F(y,t)+a_3T(y,t) \tag{7}$$

$$\frac{\partial T(y,t)}{\partial t}=a_4\frac{\partial^2T(y,t)}{\partial y^2} \tag{8}$$

where

$$a_0=\phi_1=1-\phi+\phi\frac{\rho_{\mathrm{CTN}}}{\rho_{\mathrm{f}}},\quad a_1=\phi_2\left(1+\frac{1}{\beta}\right),\quad a_2=M\phi_3,\quad a_3=\mathrm{Gr}\phi_3,\quad a_4=\frac{\lambda_{\mathrm{nf}}}{\mathrm{Pr}\,\phi_5}$$
$$M=\frac{\sigma_{\mathrm{f}}B_0^2\nu_{\mathrm{f}}}{U_0^2\rho_{\mathrm{f}}},\quad \mathrm{Gr}=\frac{\nu_{\mathrm{f}}\mathrm{g}\Delta T\beta_{\mathrm{f}}}{U_0^3},\quad \lambda_{\mathrm{nf}}=\frac{k_{\mathrm{nf}}}{k_{\mathrm{f}}},\quad \mathrm{Pr}=\frac{(\mu c_p)_{\mathrm{f}}}{k_{\mathrm{f}}}$$
$$\phi_2=\frac{1}{(1-\phi)^{2.5}},\quad \phi_3=1+\frac{3\phi(\sigma_{\mathrm{CNT}}-1)}{(2+\sigma_{\mathrm{CNT}})-\phi(\sigma_{\mathrm{CNT}}-1)}$$
$$\phi_4=1-\phi+\frac{\phi(\rho\beta_T)_{\mathrm{CTN}}}{(\rho\beta_T)_{\mathrm{f}}},\quad \phi_5=1-\phi+\frac{\phi(\rho c_p)_{\mathrm{CNT}}}{(\rho c_p)_{\mathrm{f}}} \tag{9}$$

The initial and boundary conditions become:

$$F(y,0)=0,\quad T(y,0)=0,\quad y\geq 0,\quad F(0,t)=At^{p},\quad T(0,t)=1,\quad t>0,$$
$$F(\infty,t)=0,\quad T(\infty,t)=0,\quad t>0 \tag{10}$$

Calculation of problem

Calculation of temperature and velocity fields via AB fractional derivatives

For developing AB fractional model, the governing eqs. (7) and (8) have been replaced by time variable of order α eqs. (7) and (8) are formulated in terms of AB fractional operator:

$$^{\mathrm{AB}}\left[\frac{\partial^{\alpha}F(y,t)}{\partial t^{\alpha}}\right]-\frac{a_1}{a_0}\frac{\partial^2 F(y,t)}{\partial y^2}+\frac{a_2}{a_0}F(y,t)+\frac{a_3}{a_0}T(y,t)=0 \tag{11}$$

$$^{\mathrm{AB}}\left[\frac{\partial^{\alpha}T(y,t)}{\partial t^{\alpha}}\right]-a_4\frac{\partial^2 T(y,t)}{\partial y^2}=0 \tag{12}$$

where $^{\mathrm{AB}}\left[\dfrac{\partial^{\alpha}F(y,t)}{\partial t^{\alpha}}\right]$ is AB fractional differential operator of order α defined [57-60]:

$$^{\mathrm{AB}}\left[\frac{\partial^{\alpha}F(y,t)}{\partial t^{\alpha}}\right]=\frac{M(\alpha)}{1-\alpha}\int_0^t F'(y,t)\mathbf{E}_{\alpha}\left[\frac{-\alpha(z-t)^{\alpha}}{1-\alpha}\right]\mathrm{d}t \tag{13}$$

Here $M(\alpha)$ is normalization function such that $M(0)=M(1)=1$. Employing Laplace transform on eqs. (11) and (12) and imposing assumed conditions ($10_{1,2,3}$), taking $\chi = 1/(1-\alpha)$, we obtain:

$$\left[\frac{\partial^2}{\partial y^2}-\frac{a_0\,\chi\, s^{\alpha}}{a_1\,(s^{\alpha}+\alpha\chi)}-\frac{a_2}{a_1}\right]\bar{F}(y,s)=\frac{a_3}{a_1}\bar{T}(y,s) \tag{14}$$

$$\left[\frac{\partial^2}{\partial y^2}-\frac{s^{\alpha}\chi}{a_4(s^{\alpha}+\alpha\chi)}\right]\bar{T}(y,s)=0 \tag{15}$$

where $\bar{F}(y,s)$ and $\bar{T}(y,s)$ are the Laplace transform of $F(y, t)$ and $T(y, t)$, computing eqs. (14) and (15), we arrive:

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}\exp\left[-y\sqrt{\frac{a_0\,\chi\, s^{\alpha}}{a_1\,(s^{\alpha}+s_2)}-\frac{a_2}{a_1}}\right]+\frac{a_3\,(s^{\alpha}+s_2)}{s(s_3 s^{\alpha}-s_4)}\exp\left[-y\sqrt{\frac{s_1\, s^{\alpha}}{(s^{\alpha}+s_2)}}\right] \tag{16}$$

$$\bar{T}(y,s)=\frac{1}{s}\exp\left[-y\sqrt{\frac{s_1\, s^{\alpha}}{(s^{\alpha}+s_2)}}\right] \tag{17}$$

where $s_1=\chi/a_4$, $s_2=a\chi$, $s_3=a_0a_4\chi-a_2\,a_4+a_1\chi$, $s_4=a_2a_4\alpha\chi$.

In order to obtain velocity and temperature profiles, we write eqs. (16) and (17) into series form, we get equivalent expressions:

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}+\frac{Ap!}{s^{p+1}}\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_5}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_6}{s_5}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{1}{p_3!}(-s_2)^{p_3}\sum_{p_4=0}^{\infty}\left(-\frac{s_3}{s_4}\right)^{p_4}\cdot$$

$$\cdot\frac{\Gamma\left(1+\dfrac{p_1}{2}\right)\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{\Gamma\left(\dfrac{p_1}{2}-p_2+1\right)\Gamma\dfrac{p_1}{2}s^{\alpha p_2+\alpha p_3-\alpha p_4}}-\frac{a_3}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{s_4}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_2}\right)^{p_3}\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{p_3!\Gamma\dfrac{p_1}{2}s^{\alpha p_3-\alpha p_2-\alpha+1}}-$$

$$-\frac{a_3 s_2}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{s_4}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_2}\right)^{p_3}\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{p_3!\Gamma\left(\dfrac{p_1}{2}\right)s^{\alpha p_3-\alpha p_2-\alpha+1}} \tag{18}$$

$$\bar{T}(y,s)=\frac{1}{s}+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\frac{(-s_2)^{p_2}\Gamma\left(p_2+\dfrac{p_1}{2}\right)}{p_2!\Gamma\dfrac{p_1}{2}s^{\alpha p_2+1}} \tag{19}$$

where $s_3 = a_0\chi - a_2$, $s_6 = -a_2s_2$. Inverting eqs. (18) and (19) *via* Laplace transform and expressing the final solutions in terms of newly defined **M**-function $\mathbf{M}_q^p(F)$, and Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^y(T)$, respectively:

$$F(y,t)=At^p+Ap!\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_5}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_6}{s_5}\right)^{p_2}\sum_{p_4=0}^{\infty}\left(-\frac{s_3}{s_4}\right)^{p_4}\cdot$$

$$\cdot\mathbf{M}_3^2\left[(-s_2)\left|\begin{matrix}\left(\dfrac{p_1}{2}+1,0\right), & \left(\dfrac{p_1}{2},1\right) & \\ \left(\dfrac{p_1}{2},0\right), & \left(\dfrac{p_1}{2}-p_2+1,0\right), & (\alpha p_2-\alpha p_4-1,\alpha)\end{matrix}\right.\right]-$$

$$-\frac{a_3}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{s_4}\right)^{p_2}\mathbf{M}_2^1\left[(-s_2)\left|\begin{matrix}\left(\dfrac{p_1}{2},1\right) & \\ \left(\dfrac{p_1}{2},0\right), & \left(-\alpha p_2-\alpha,\alpha\right)\end{matrix}\right.\right]-$$

$$-\frac{a_3 s_2}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1\,!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{s_4}\right)^{p_2}\mathbf{M}_2^1\left[(-s_2)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\quad(-\alpha p_2-\alpha,\alpha)\end{matrix}\right.\right] \tag{20}$$

$$T(y,t)=1+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1\,!}\mathbf{M}_{p_2,1}^{\frac{p_1}{2}}(-s_2 t) \tag{21}$$

where the newly defined Generalized **M**-function $\mathbf{M}_q^p(F)$ [61-65] and Mittage Leffler function $\mathbf{M}_{\varepsilon,\delta}^y(T)$ are described, respectively:

$$t^{\varepsilon_q-1}\sum_{\delta}^{\infty}\frac{(F)^{\delta}\prod_{h=1}^{f}\Gamma(a_h+A_h\delta)}{\delta!\prod_{h=1}^{g}\Gamma(b_h+B_h\delta)}=\mathbf{M}_q^p\left[F\left|\begin{matrix}(a_1,A_1),(a_2,A_2),...,(a_f,A_f)\\(b_1,B_1),(b_2,B_2),...,(b_g,B_g)\end{matrix}\right.\right] \tag{22}$$

$$t^{\delta-1}\sum_{\zeta}^{\infty}\frac{(T)^{\zeta}\Gamma(y+\zeta)}{\zeta!\Gamma(y)\Gamma(\varepsilon\zeta+\delta)}=t^{\delta-1}\mathbf{E}_{\varepsilon,\delta}^y(T)=\mathbf{M}_{\varepsilon,\delta}^y(T),\qquad \mathrm{Re}(\varepsilon)>0,\quad \mathrm{Re}(\delta)>0 \tag{23}$$

Calculation of temperature and velocity fields via CF fractional derivatives

For developing CF fractional model, the governing eqs. (7) and (8) have been replaced by time variable of order β eqs. (7) and (8) are formulated in terms of CF fractional operator:

$${}^{\mathrm{CF}}\left[\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right]-\frac{a_1}{a_0}\frac{\partial^2 F(y,t)}{\partial y^2}+\frac{a_2}{a_0}F(y,t)+\frac{a_3}{a_0}T(y,t)=0 \tag{24}$$

$${}^{\mathrm{CF}}\left[\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right]-a_4\frac{\partial^2 T(y,t)}{\partial y^2}=0 \tag{25}$$

where ${}^{\mathrm{CF}}\left[\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right]$ is the CF fractional differential operator of order β defined [66-68]:

$${}^{\mathrm{CF}}\left[\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right]=\frac{M(\beta)}{1-\beta}\int_0^t F'(y,t)\mathrm{Exp}\left[\frac{-(t-\delta)\beta}{(1-\beta)}\right]\mathrm{d}t \tag{26}$$

where $M(\beta)$ is the normalization function such that $M(0) = M(1) = 1$. Employing Laplace transform on eqs. (24) and (25) and imposing assumed conditions ($10_{1,2,3}$), taking $\tau = 1/(1 - \beta)$, we obtain:

$$\left[\frac{\partial^2}{\partial y^2}-\frac{a_0\tau s}{a_1(s+\beta\tau)}-\frac{a_2}{a_1}\right]\bar{F}(y,s)=\frac{a_3}{a_1}\bar{T}(y,s) \tag{27}$$

$$\left[\frac{\partial^2}{\partial y^2}-\frac{s\tau}{a_4\left(s+\tau\beta\right)}\right]\bar{T}(y,s)=0 \tag{28}$$

where $\bar{F}(y,s)$ and $\bar{T}(y,s)$ are the Laplace transform of $F(y, t)$ and $T(y, t)$, computing eqs. (27) and (28), we arrive:

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}\exp\left[-y\sqrt{\frac{a_0\tau s}{a_1(s+s_8)}-\frac{a_2}{a_1}}\right]+\frac{a_3(s+s_8)}{s(s_9 s-s_{10})}\exp\left[-y\sqrt{\frac{s_7 s}{(s+s_8)}}\right] \tag{29}$$

$$\bar{T}(y,s)=\frac{1}{s}\exp\left[-y\sqrt{\frac{s_7 s}{(s+s_8)}}\right] \tag{30}$$

where $s_7 = \tau/a_4$, $s_8 = \beta\tau$, $s_9 = a_0a_4\tau - a_2a_4 + a_1\tau$, $s_{10} = a_2a_4\beta\tau$. In order to obtain velocity and temperature profiles, we write eqs. (29) and (30) into series form, we get equivalent expressions:

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}+\frac{Ap!}{s^{p+1}}\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_{11}}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_{12}}{s_{11}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{1}{p_3!}\left(-s_8\right)^{p_3}\sum_{p_4=0}^{\infty}\left(-\frac{s_9}{s_{10}}\right)^{p_4}\cdot$$

$$\cdot\frac{\Gamma\left(1+\dfrac{p_1}{2}\right)\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{\Gamma\left(\dfrac{p_1}{2}-p_2+1\right)\Gamma\left(\dfrac{p_1}{2}\right)s^{p_2+p_3-p_4}}-\frac{a_3}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_8}\right)^{p_3}\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{p_3!\Gamma\left(\dfrac{p_1}{2}\right)s^{p_3-p_2}}-$$

$$-\frac{a_3 s_8}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_8}\right)^{p_3}\Gamma\left(p_3+\dfrac{p_1}{2}\right)}{p_3!\Gamma\left(\dfrac{p_1}{2}\right)s^{p_3-p_2}} \tag{31}$$

$$\bar{T}(y,s)=\frac{1}{s}+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\frac{\left(-s_8\right)^{p_2}\Gamma\left(p_2+\dfrac{p_1}{2}\right)}{p_2!\Gamma\left(\dfrac{p_1}{2}\right)s^{p_2+1}} \tag{32}$$

where $s_{11} = a_0\tau - a_2$, $s_{12} = - a_2 s_8$. Inverting eqs. (31) and (32) *via* Laplace transform and expressing the final solutions in terms of newly defined **M**-function $\mathbf{M}_q^p(F)$ and Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^y(T)$, respectively:

$$F(y,t) = At^p + Ap!\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_{11}}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_{12}}{s_{11}}\right)^{p_2}\sum_{p_4=0}^{\infty}\left(-\frac{s_9}{s_{10}}\right)^{p_4}\cdot$$

$$\cdot\mathbf{M}_3^2\left[\left(-s_8\right)\left|\begin{matrix}\left(\frac{p_1}{2}+1,0\right),\left(\frac{p_1}{2},1\right)\\\left(\frac{p_1}{2},0\right),\left(\frac{p_1}{2}-p_2+1,0\right),\left(p_2-p_4-1,1\right)\end{matrix}\right.\right]-$$

$$-\frac{a_3}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\mathbf{M}_2^1\left[\left(-s_8\right)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\\left(\frac{p_1}{2},0\right),\left(-p_2-1,1\right)\end{matrix}\right.\right]-$$

$$-\frac{a_3 s_8}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\mathbf{M}_2^1\left[\left(-s_8\right)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\\left(\frac{p_1}{2},0\right),\left(-p_2-1,1\right)\end{matrix}\right.\right] \tag{33}$$

$$T(y,t) = 1 + \sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\mathbf{M}_{p_2,1}^{\frac{p_1}{2}}(-s_8 t) \tag{34}$$

Results and discussion

This portion is fascinated to a comprehensive study of single- and multi-WCNT with all pertinent parameters on velocity field and temperature distribution. Heat transfer analysis of SWCNT and MWCNT for electrically conducting flow of Casson fluid is carried out. Single- and multi-WCNT are suspended in methanol that is considered as a conventional base fluid. The governing PDE of nanofluids have been modeled by AB and CF fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established *via* both approaches *i. e.* AB and CF fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^y(T)$ and generalized **M**-function $\mathbf{M}_q^p(F)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases (AB and CF fractional derivatives) are depicted for graphical illustrations and the comparison of three types of fluids: pure methanol, methanol with SWCNT, and methanol with MWCNT is portrayed *via* AB and CF fractional derivatives. However, the major outcomes and consequences are expected as:

the general solutions are investigated by introducing AB and CF fractional derivatives and presented in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}(T)$ and generalized $\mathbf{M}$-function $\mathbf{M}_{q}^{p}(F)$ which satisfy imposed conditions. Figures 2(a) and 2(b) elucidate the effects of nanoparticles volume fraction on velocity field and temperature distribution. It is apparent from fig. 2(a) that velocity fields investigated *via* AB and CF approach are increasing function with increasing nanoparticles volume fraction. On the contrary in fig. 2(b), temperature field is decreasing by the variation of nanoparticles volume fraction through CF approach. Meanwhile, AB approach has increasing behavior, both approaches have opposite trend as well. It is in accordance with the physical expectation that when the thermal conductivity is raised up then temperature of fluid is enlarged. Figure 3 is depicted to elaborate the characteristics of transverse magnetic field which results the flow of velocity field in reciprocating influences. In brevity, the velocity field obtained by AB approach generates decelerations in fluid flow. The CF approach has reversal behavior in comparison with AB approach. It is also noted that the fluid flow is sequestrating and scattering over the whole vicinity of plate reciprocally. Figure 4 is prepared for comparison of analytical solutions established by AB and CF approaches. The comparison of three types of nanofluids namely: pure methanol, methanol with SWCNT, and methanol with MWCNT is underlined by both fractional derivatives.

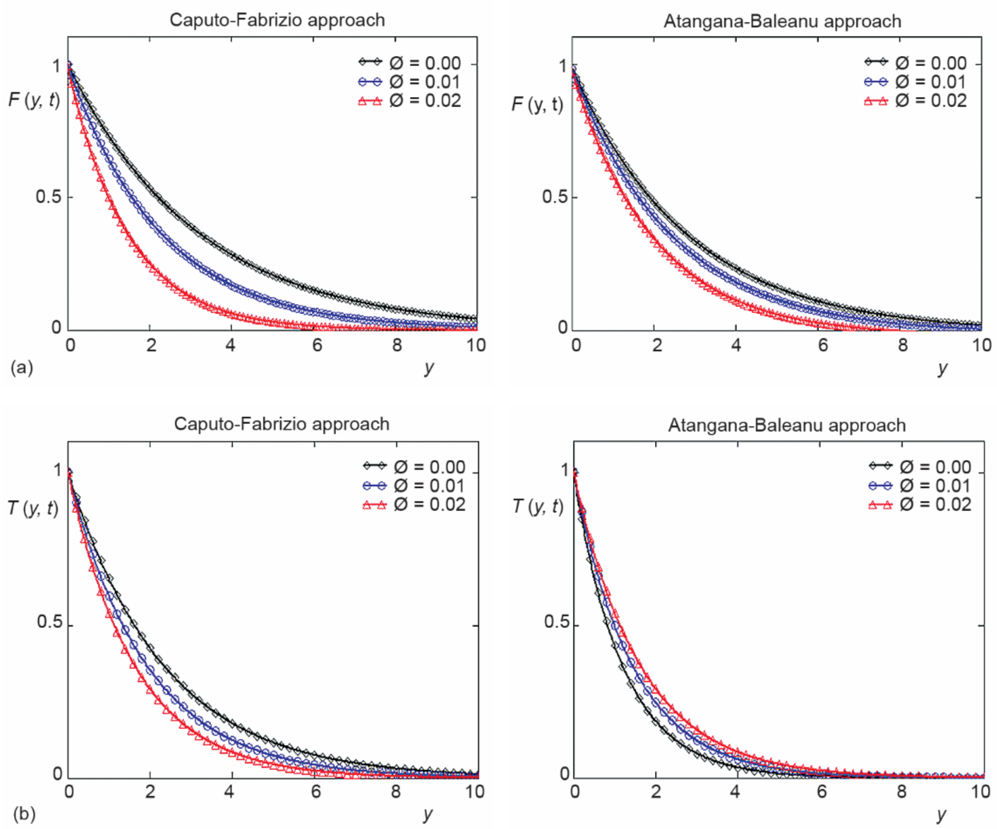

Figure 2. (a) Profile of velocity field *via* CF and AB fractional operators for nanoparticles volume fraction, (b) profile of temperature distribution *via* CF and AB fractional operators for nanoparticles volume fraction

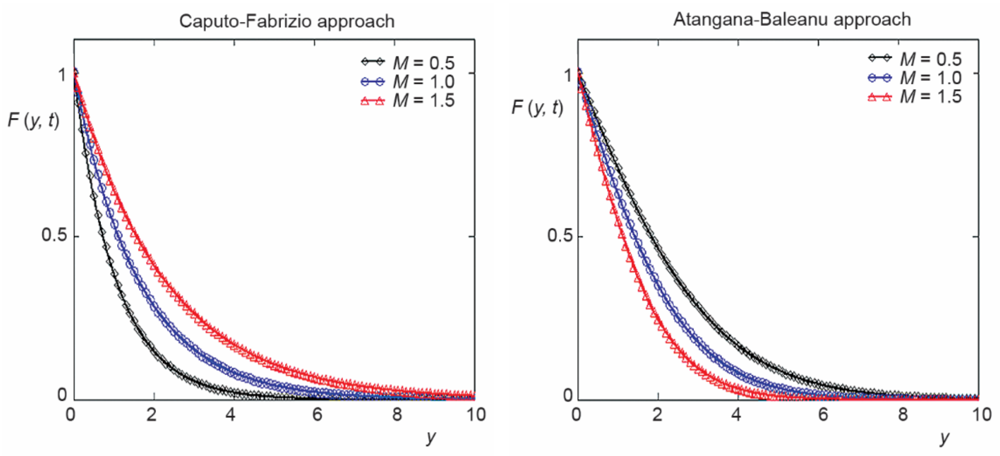

Figure 3. Profile of velocity field *via* CF and AB fractional operators for magnetic field

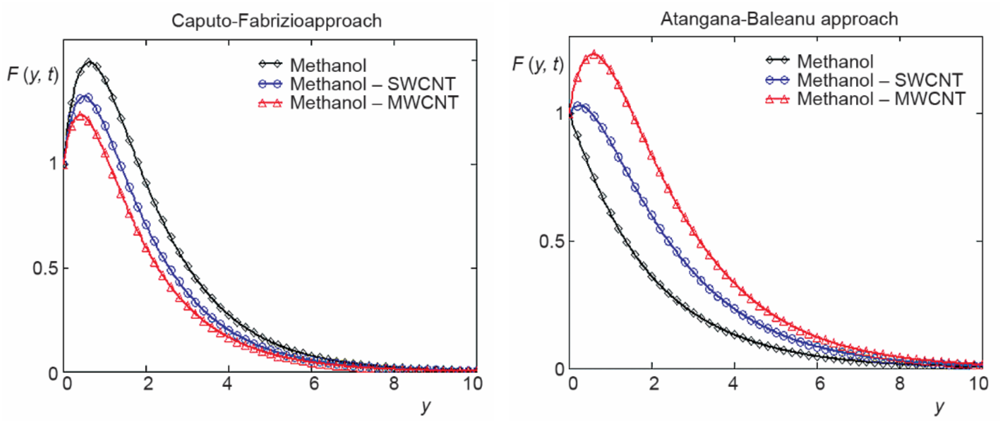

Figure 4. Comparison of velocity field *via* CF and AB fractional operators for three types of models

Conclusions

Applying newly defined fractional approaches (derivatives) namely Atangana-Baleanu and Caputo-Fabrizio fractional derivatives on the governing PDE of nanofluids, the comparison of analytical solutions for temperature distribution and velocity field has been investigated. The general analytical solutions have been obtained via Laplace transform and expressed in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}(T)$ and generalized $\mathbf{M}$-function $\mathbf{M}_{q}^{p}(F)$ satisfying initial and boundary conditions. The major findings are listed below for the vivid rheological effects which are summarized as follows.

- The velocity field is investigated with AB approach has opposite trend of fluid flow than the velocity field with CF approach. It is also noted that the velocity field with both approaches tends to nearer and nearer over the whole domain of plate.
- The characteristics of transverse magnetic field results the flow of velocity field in reciprocating influences. Simply, the velocity field obtained by AB approach generates decelerations in fluid flow. On the contrary, CF approach has reversal behavior in comparison with AB approach.

- The effects of nanoparticles volume fraction are investigated for temperature distribution as well as velocity field *via* AB and CF approaches are increasing and decreasing function, respectively.
- The comparison of analytical solutions by AB and CF approaches is also performed for three types of nanofluids. Here, pure methanol moves quicker in comparison with methanol-SWCNT and methanol-MWCNT. While methanol-MWCNT moves more rapidly in comparison with pure methanol and methanol-SWCNT. This leads to the phenomenon that MWCNT have slighter thermal conductivity as well as density than SWCNT.

Table 1. Thermo-physical properties of Methanol and nanoparticles (CNT) [51, 52]

Base fluid/nanoparticles	ρ [kgm^{-3}]	C_p [Jkg^{-1}K^{-1}]	k [Wm^{-1}K^{-1}]
Methanol	792	2545	0.2035
SWCNT	2600	425	6600
SWCNT	1600	796	3000

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