

THE STUDY OF HEAT TRANSFER PHENOMENA BY USING MODIFIED HOMOTOPY PERTURBATION METHOD COUPLED BY LAPLACE TRANSFORM

by

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In this paper, we present modified homotopy perturbation method coupled by Laplace transform to solve non-linear problems. As case study modified homotopy perturbation method coupled by Laplace transform is employed in order to obtain an approximate solution for the non-linear differential equation that describes the steady-state of a heat 1-D flow. The comparison between approximate and exact solutions shows the practical potentiality of the method.

Key words: homotopy perturbation method, Laplace transform, heat conduction

Introduction

The heat transfer laws are of paramount importance in the design and operation of equipment in many industrial applications as well as in pure sciences. Therefore, it is important to search for accurate analytical approximate solutions for the equations describing these phenomena. However, it is well known that non-linear differential equations that describe them are difficult to solve [1-3].

Laplace transform (LT) plays a relevant role in mathematics because its methods allow to solve many problems in science and engineering [4]. It is well known that LT is a powerful tool useful for solving linear ODE with constant coefficients and initial conditions and to solve some cases of differential equations with variable coefficients and PDE [4]. The contribution of LT to non-linear ODE has required its combination with other techniques. Thus, [1] reported a combination of homotopy perturbation method (HPM) and LT methods (LT-HPM) in order to obtain highly accurate approximate solutions for these equations. At this time, it is clarified that the coupling of LT and HPM is known by another name in the literature: A modified homotopy perturbation method coupled by Laplace transform (MHPMLT) [5-7]. As a matter of fact, [5]

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named this modification as He-Laplace method for simplicity. Thus, from here on, we agree to call the proposed method MHPMLT. This work proposes MHPMLT method in the search for approximate solutions for the non-linear ODE with Dirichlet boundary conditions defined on a finite interval [8] which describes the steady-state 1-D heat conduction in a slab with temperature-dependent thermal conductivity [9]. The case of equations with boundary conditions on infinite intervals has been reported for some authors [10, 11] and correspond often to problems defined on semi-infinite ranges. However, the methods for solving these problems are different from those that will be presented in this study [8]. Non-linear problems frequently arise in science and engineering, whereby, it is very important to search on differential equations that describe them. In recent years, there have been proposed several methods focused to find approximate solutions to non-linear differential equations; such as those based on: variational approaches [12], tanh method [13], exp-function [14], Adomian's decomposition method (ADM) [15], parameter expansion [16], homotopy analysis method (HAM) [2, 3], perturbation method [17], and HPM [1, 5-11, 18-23], since the main solution process of this article is HPM, next we briefly mention some of the last developments of this method; such as the coupling of HPM and Frobenius method [20], multiple scales HPM method [21], parametrized HPM [22], non-linearities distribution HPM used to find the solution of Troesch problem [23], among many others.

Standard HPM

The standard HPM was proposed by He, it was introduced like a powerful tool to approach various kinds of non-linear problems. The HPM is considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to small parameters as occur with traditional perturbation methods. For example, HPM method requires neither small parameter nor linearization, but only few iterations to obtain highly accurate solutions [19, 24].

To figure out how HPM works, consider a general non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (1)$$

with the following boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Omega \quad (2)$$

where A is a general differential operator, B – the a boundary operator, and $f(r)$ – the known analytical function. The A can be divided into two operators L and N , where L is linear and N non-linear; so that eq. (1) can be re-written:

$$L(u) + N(u) - f(r) = 0 \quad (3)$$

Generally, a homotopy can be constructed [19, 24]:

$$H(U, p) = (1 - p)[L(U) - L(u_0)] + p[L(U) + N(U) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega \quad (4)$$

or

$$H(U, p) = L(U) - L(u_0) + p[L(u_0) + N(U) - f(r)] = 0, \quad p \in [0, 1] \quad (5)$$

where p is a homotopy parameter, whose values are within range of 0 and 1, and u_0 is the first approximation for the solution of eq. (3) that satisfies the boundary conditions.

Assuming that solution for eq. (4) or eq. (5) can be written as a power series of p :

$$U = v_0 + v_1 p + v_2 p^2 + \dots \quad (6)$$

Substituting eq. (6) into eq. (5) and equating identical powers of p terms there can be found values for the sequence v_0, v_1, v_2, \dots

When $p \rightarrow 1$ yields to the approximate solution for eq. (1) in the form:

$$U = v_0 + v_1 + v_2 + v_3 \dots \quad (7)$$

Basic idea of MHPMLT

The objective of this section is showing how MHPMLT can be employed to find analytical approximate solutions for ODE as eq. (3).

For this purpose, MHPMLT follows the same steps of standard HPM until eq. (5), next we apply LT on both sides of homotopy equation eq. (5):

$$y(z) = 1 - 0.7709169974z - 0.2290830032z^3 \quad (8)$$

using the differential property of LT, we have [4]:

$$\begin{aligned} s^n \mathfrak{L}\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \dots - U^{(n-1)}(0) = \\ = \mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\} \end{aligned} \quad (9)$$

or

$$\begin{aligned} \mathfrak{L}(U) = \left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \\ + \left(\frac{1}{s^n}\right)\mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\} \end{aligned} \quad (10)$$

applying inverse Laplace transform to both sides of eq. (10):

$$U = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \left(\frac{1}{s^n}\right)\mathfrak{L}\{L(u_0) - pL(u_0) + p[-N(U) + f(r)]\}\right\} \quad (11)$$

Assuming that solutions of eq. (3) can be expressed as a power series of p :

$$U = \sum_{n=0}^{\infty} p^n v_n \quad (12)$$

then substituting eq. (12) into eq. (11), we get:

$$\sum_{n=0}^{\infty} p^n v_n = \mathfrak{L}^{-1}\left\{\left(\frac{1}{s^n}\right)\{s^{n-1}U(0) + s^{n-2}U'(0) + \dots + U^{(n-1)}(0)\} + \mathfrak{L}\{L(u_0) - pL(u_0) + p[-N\left(\sum_{n=0}^{\infty} p^n v_n\right) + f(r)]\}\right\} \quad (13)$$

comparing coefficients of p , with the same power leads to:

$$\begin{aligned}
p^0 : v_0 &= \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^n} \right) \left(s^{n-1} U(0) + s^{n-2} U'(0) + \dots + U^{(n-1)}(0) \right) + \mathfrak{I} \{ L(u_0) \} \right\} \\
p^1 : v_1 &= \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^n} \right) \left(\mathfrak{I} \{ -N(v_0) - L(u_0) + f(r) \} \right) \right\} \\
p^2 : v_2 &= \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^n} \right) \mathfrak{I} \{ -N(v_0, v_1) \} \right\} \\
p^3 : v_3 &= \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^n} \right) \mathfrak{I} \{ -N(v_0, v_1, v_2) \} \right\} \\
&\dots \\
p^j : v_j &= \mathfrak{I}^{-1} \left\{ \left(\frac{1}{s^n} \right) \mathfrak{I} \{ -N(v_0, v_1, v_2, \dots, v_j) \} \right\} \\
&\dots
\end{aligned} \tag{14}$$

Assuming that the initial approximation has the form: $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1, \dots, U^{(n-1)}(0) = \alpha_{n-1}$; then the approximate solution may be obtained:

$$u = \lim_{p \rightarrow 1} U = v_0 + v_1 + v_2 + \dots \tag{15}$$

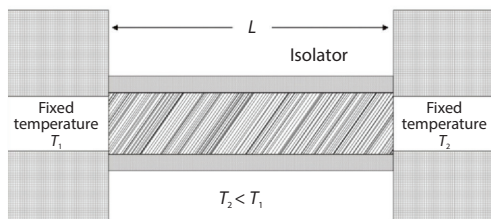


Figure 1. The 1-D conduction of heat through an insulated slab

Governing equations

The goal of this work is the searching for an approximate solution for the non-linear problem, which describes the steady-state 1-D conduction of heat in a slab with thermal conductivity linearly dependent on the temperature, see fig. 1.

The transferred energy caused by the temperature difference between two adjacent parts of a body is called heat conduction [25]. Let $u(x, y, z, t)$ be the temperature of the aforementioned slab at a point (x, y, z) at time t , and K , σ , and μ the thermal conductivity, specific heat, and density of the solid, respectively; then it is verified that the temperature obeys the following partial differential equation [26]:

$$\frac{\partial}{\partial x} \left(K \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(K \frac{\partial u}{\partial z} \right) = \sigma \mu \frac{\partial u}{\partial t} \tag{16}$$

which is known as the heat conduction equation.

In the cases where K is a constant, the aforementioned equation reduces to the following known linear partial differential equation:

$$K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \sigma \mu \frac{\partial u}{\partial t} \tag{17}$$

A substance with a high thermal conductivity is a good heat conductor; on the contrary, one with a small thermal conductivity is a poor conductor of heat, or equivalently a good thermal insulator. The K -value depends on the temperature, increasing slightly when it increases, but can be considered almost constant throughout a substance if the temperature difference between the parts is not too large [25].

In the case of heat flow under steady conditions, the temperature does not depend on time, t , such that $\partial u / \partial t = 0$. Thus eq. (16) becomes:

$$\frac{\partial}{\partial x} \left(K \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(K \frac{\partial u}{\partial z} \right) = 0 \quad (18)$$

while eq. (17) becomes in the Laplace equation for u :

$$\nabla^2 u = 0 \quad (19)$$

Solutions to the linear heat conduction equations for constant thermal conductivity eqs. (17) and (19) are studied in detailed, for instance in [25, 26]. Nonetheless unlike the aforementioned, in general K is dependent on temperature, and in this case eqs. (16) and (18), are non-linear.

This article considers the case of steady conditions for the 1-D conduction of heat in a slab of thickness, L , assuming a temperature dependent thermal conductivity K [9]. Supposing that the temperatures of the two opposite faces of the slab are uniformly maintained at T_1 and T_2 , where $T_2 < T_1$. Then, the governing equation is obtained as a 1-D case of eq. (18):

$$\frac{d}{dx} \left(K \frac{du}{dx} \right) = 0 \quad (20)$$

subject to the following boundary conditions:

$$u(0) = T_1, \quad u(L) = T_2 \quad (21)$$

For the sake of simplicity, we assume that the thermal conductivity varies linearly with temperature, thus [9] (see discussion):

$$K = K_2 [1 + \beta(u - T_2)] \quad (22)$$

where K_2 is the thermal conductivity at temperature T_2 and the constant β is defined.

In order to employ MHPMLT to obtain a handy accurate analytical approximate solution for the heat problem previously mentioned, we rewrite eq. (20) as follows. After performing the indicated derivative in eq. (20):

$$K \frac{d^2 u}{dx^2} + \frac{dK}{dx} \frac{du}{dx} = 0 \quad (23)$$

Next, it is suggested the introduction of the following dimensionless quantities [9]:

$$y = \frac{u - T_2}{T_1 - T_2}, \quad z = \frac{x}{L}, \quad \varepsilon = \beta(T_1 - T_2) = \frac{K_1 - K_2}{K_2} \quad (24)$$

note that the last equality of eq. (24) defines β .

Employing the chain rule, it is possible to deduce:

$$\frac{d}{dx} = \frac{1}{L} \frac{d}{dz}, \quad \frac{d^2}{dx^2} = \frac{1}{L^2} \frac{d^2}{dz^2} \quad (25)$$

On the other hand, the first equation of eq. (24) can be written:

$$u = T_2 + y(T_1 - T_2) \quad (26)$$

Thus, substituting eq. (26) into eq. (22), yields:

$$K = K_2 [1 + \varepsilon y] \quad (27)$$

After substituting eqs. (25)-(27) into eq. (23):

$$\frac{d^2 y}{dz^2} + \varepsilon \frac{d^2 y}{dz^2} + \varepsilon \left(\frac{dy}{dz} \right) = \quad (28)$$

where boundary conditions eq. (21) adopt the simpler form:

$$y(0) = 1, \quad y(1) = 0 \quad (29)$$

Case study

The objective of this section is employ MHPMLT, to find an analytical approximate solution for the non-linear problem given by eqs. (28) and (29).

We will see that it is possible to find a handy solution by applying MHPMLT method. Identifying terms:

$$L(y) = y''(z) \quad (30)$$

$$N(y) = \varepsilon[y''(z)y(z) + y'^2(z)] \quad (31)$$

where prime denotes differentiation respect to z .

In accordance with eq. (4), we propose:

$$(1-p)(y'' - y_0'') + p[y'' + \varepsilon y y'' + \varepsilon y'^2] = 0 \quad (32)$$

or

$$y'' = y_0'' + p[-y_0'' - \varepsilon y y'' - \varepsilon y'^2] \quad (33)$$

Applying LT we get:

$$s^2 Y(s) - A - s = \mathfrak{I} \left[y_0'' + p(-y_0'' - \varepsilon y y'' - \varepsilon y'^2) \right] \quad (34)$$

where we have defined $Y(s) = \mathfrak{I}[y(z)]$, $A = y'(0)$ with initial condition $y(0) = 1$.

Solving for $Y(s)$ and applying Laplace inverse transform \mathfrak{I}^{-1} :

$$y(z) = \mathfrak{I}^{-1} \left\{ \frac{1}{s} + \frac{A}{s^2} + \frac{1}{s^2} \mathfrak{I} \left[y_0'' + p(-y_0'' - \varepsilon y y'' - \varepsilon y'^2) \right] \right\} \quad (35)$$

Next, we assume a series solution for $y(z)$, in the form:

$$y(z) = \sum_{n=0}^{\infty} p^n v_n(z) \quad (36)$$

where

$$v_0(z) = 1 + Az \quad (37)$$

is chosen as the first approximation for the solution of (28) that satisfies the conditions $y(0) = 1$, $y'(0) = A$.

Substituting eqs. (36) and (37) into eq. (35), we get:

$$\begin{aligned} \sum_{n=0}^{\infty} p^n v_n = \mathfrak{I}^{-1} \left\{ \frac{1}{s} + \frac{A}{s^2} + \frac{1}{s^2} \mathfrak{I} \left(y_0'' + p \left(-y_0'' - \varepsilon (v_0 + p v_1 + p^2 v_2 + \dots) (v_0'' + p v_1'' + p^2 v_2'' + \dots) \right) \right) \right\} - \\ - \varepsilon \mathfrak{I}^{-1} \left\{ \frac{1}{s^2} \mathfrak{I} \left((v_0' + p v_1' + p^2 v_2' + \dots)^2 \right) \right\} \end{aligned} \quad (38)$$

On comparing the coefficients of like powers of p :

$$p^0 : v_0(z) = \mathfrak{I}^{-1} \left\{ \frac{1}{s} + \frac{A}{s^2} \right\} \quad (39)$$

$$p^1 : v_1(z) = \mathfrak{I}^{-1} \left\{ \left(\frac{-\varepsilon}{s^2} \right) \mathfrak{I} \{ v_0 v_0'' + v_0'^2 \} \right\} \quad (40)$$

$$p^2 : v_2(z) = \mathfrak{I}^{-1} \left\{ \left(\frac{-\varepsilon}{s^2} \right) \mathfrak{I} \{ v_0 v_1'' + v_1 v_0'' + 2v_0' v_1' \} \right\} \quad (41)$$

...

After solving the LT for $v_0(z)$, $v_1(z)$, $v_2(z)$,... we obtain:

$$p^0 : v_0(z) = 1 + Az \quad (42)$$

$$p^1 : v_1(z) = \frac{-\varepsilon A^2 z^2}{2} \quad (43)$$

$$p^2 : v_2(z) = \varepsilon^2 \left[\frac{A^2 z^2}{2} + \frac{A^3 z^3}{2} \right] \quad (44)$$

...

and so on.

By substituting eqs. (42)-(44) into eq. (15) and evaluating the limit $p \rightarrow 1$, results in a handy second order approximation:

$$y(z) = 1 + Az + \frac{\varepsilon A^2}{2} (-1 + \varepsilon) z^2 + \frac{\varepsilon^2}{2} A^3 z^3 \quad (45)$$

In order to calculate the value of A , we require that eq. (45) satisfies the boundary condition $y(1) = 0$. Considering the case studies $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 1.5$, we obtain, respectively, the solutions:

$$y(z) = 1 - 0.838447674076117z - 0.0878743127704564z^2 - 0.0736780131534264z^3 \quad (46)$$

$$y(z) = 1 - 0.7709169974z - 0.2290830032z^3 \quad (47)$$

$$y(z) = 1 - 0.744115289348098z + 0.207640336440601z^2 - 0.463525047092503z^3 \quad (48)$$

It should be mentioned that the problem eqs. (20)-(21) has the exact solution (see discussion):

$$u = \frac{-(1 - \beta T_2) + \sqrt{(1 - \beta T_2)^2 + 2\beta H_1}}{\beta} \quad (49)$$

where

$$H_1 = \frac{\beta T_1^2}{2} + (1 - \beta T_2) T_1 + \frac{T_2 - T_1}{L} \left[1 - \frac{\beta}{2} (T_2 - T_1) \right] x \quad (50)$$

Discussion

This work employed MHPMLT in the search for a handy accurate analytical approximate solution for the non-linear second order ODE with finite boundary conditions, which describes the steady-state 1-D heat conduction in a slab with thermal conductivity, linearly dependent on the temperature. As it is well known, the temperature field of a body approaches asymptotically to steady-state conditions. Therefore, the knowledge of the stationary solution is relevant because it determines the final temperature distribution along the slab. The case of thermal conductivity linearly dependent on the temperature arises, for instance, in the case of a pure metal such as copper. For this metal, at a temperature ranging from 0-5 K, the thermal conductivity is modelled mainly by electrons and increases linearly with temperature, see eq. (22), [27]. For other temperature ranges, this dependency is no longer linear, and should be the subject of future investigations, using for instance MHPMLT, to model the temperature distribution in these cases [27]. From eq. (14) is inferred that MHPMLT is expressed in terms of the initial conditions for a given differential equation, therefore, our procedure was aimed to express the approximate solutions in terms of A [8], so that $y'(0)$ can be determined, just requiring that the approximate solution satisfies the boundary condition $y(1) = 0$. This condition defines an algebraic equation for A , whose solution concludes the procedure by obtaining an analytical approximate solution for the proposed problem.

Figures 2-4 show the comparison between numerical solutions and approximate solutions (46)-(48) for $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 1.5$, respectively. It can be noticed that curves are in good agreement, whereby it is inferred the potentiality of MHPMLT in the search for approximate solutions of non-linear problems with finite boundary conditions [8]. Nevertheless, in more precise terms, it is possible to verify the accuracy of our results by calculating the square residual error (SRE) of approximate solutions (46)-(48). The SRE is defined:

$$\int_a^b R^2[u(r)]dr$$

where a and b are the end points, and $u(r)$ is an approximate solution the equation be solved (3), in our case (28) [9] and the residual $R[u(r)]$ results of substituting $u(r)$ into differential equation be solved. The resulting values were, respectively of 0.0024, 0.0747, and 0.6745 which confirms the accuracy of the proposed solutions. The SRE is in general terms a positive number, representative of the total error committed, by using the approximate solution $u(r)$ [9].

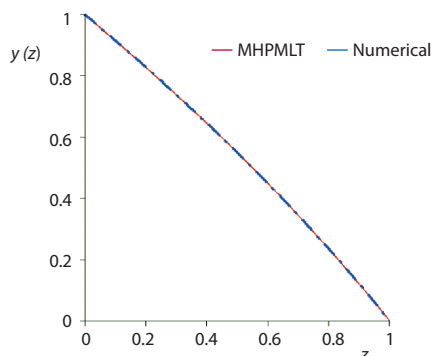


Figure 2. Comparison numerical solution of the non-linear problem given by eqs. (28) and (29) for and MHPMLT approximation eq. (46)

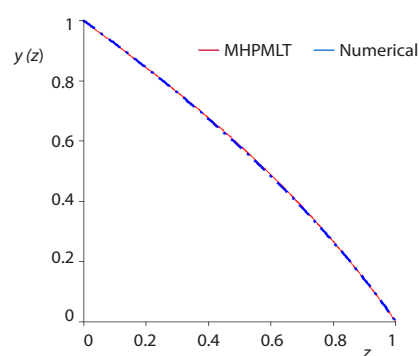


Figure 3. Comparison numerical solution of the non-linear problem given by eqs. (28) and (29) for and MHPMLT approximation eq. (47)

The parameter ε , turns out to be of paramount importance for our study. Thus, ε would be small for the case of small difference of temperature of the two opposite faces of the slab, see eq. (24) and for the same reason, for two given arbitrary points on the slab, this limit corresponds to the simpler case in which, thermal conductivity is almost constant, see eq. (22). From eq. (28), it is deduced that for steady conditions, temperature varies linearly with z in this limit.

A more interesting case would occur if the temperature gradient along the slab was not necessarily small and corresponded to bigger values of ε . From eq. (28), it is clear that the non-linear term becomes important too. A relevant fact from MHPMLT follows from equations like (28), which can be written in the form $L(z) + \varepsilon N(z) = 0$, where $L(z)$ is linear and $N(z)$ non-linear [28]. It is well known that classical methods of approximation as perturbation method (PM) [28] provides in general, better results for small perturbation parameters $\varepsilon \ll 1$. From this point of view, ε can be visualized as a parameter of smallness that measures how greater is the contribution of linear term $L(z)$ than the one of $N(z)$. In general, it is easier to find analytical approximate solutions for small values of ε than for big values of the same. Figures 2-4 and the values of square residual error obtained show a noticeable fact, that eqs. (47) and (48) provide a good approximation as solutions of eq. (28), despite of the fact that perturbation parameters $\varepsilon = 1$ and $\varepsilon = 1.5$ are indeed large. Thus, in principle, the proposed methodology is not restricted to small parameters [8] and it is able to explain the phenomenon under study for a wide range of values of the aforementioned parameter in a simple way.

On the other hand, it is very important to emphasize that, it is possible to improve the accuracy of our results, see fig. 4, considering higher order approximate solutions.

In [9] optimal homotopy perturbation method (OHPM) was employed in order to provide an approximate solution for eq. (28). Although the solution reported is handy and has good accuracy, this method is more complicated for applications than MHPMLT. The OHPM requires construct a homotopy, which includes the presence of certain functions, provided of some parameters which are determined in order to control the convergence of solutions. This procedure is usually longer and difficult than MHPMLT which many times requires only of calculating elementary LT in a systematic way.

At difference of other methods (for instance PM) which include the boundary conditions from the beginning of the problem at the lowest order approximation, MHPMLT estimates one of the initial conditions unknown at first, requiring that the whole proposed solution satisfies one of the boundary conditions (the other boundary condition is satisfied from the beginning of the procedure) thus, it is ensured that the approximate solution fits correctly on both boundaries of the interval (the above is provided, by calculating the value of A) [8].

Although this case study admitted the exact solution (49), it is necessary to make the following observations. We note that eqs. (49) and (50) provide a solution long and somehow cumbersome for practical applications. Even though eqs. (46)-(48) are only approximate solutions for the propose problem, they are handy, accurate and for the same reason ideal for practical applications. Other theoretical and practical reasons related with the aforementioned, in favor of

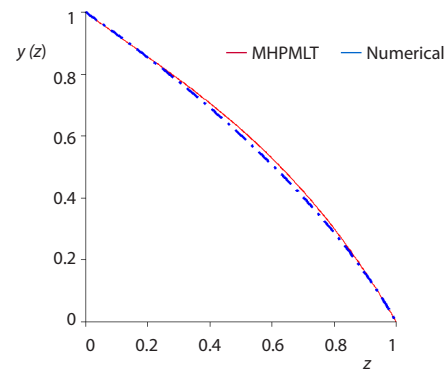


Figure 4. Comparison numerical solution of the non-linear problem given by eqs. (28) and (29) for and MHPMLT approximation eq. (48)

using the proposed method is that, the second order approximation (45) provided by MHPMLT is not only handy but it is expressed in terms of the perturbation parameter, ε . Thus, from previously explained, at least for the case of moderated temperature gradients it is possible to employ (45) instead of the rather cumbersome and complicated exact solution (49) with small loss of precision, (45) is expressed in terms of physical parameter, ε which allows in principle estimate the contributions of the different ε power terms. For example if ε is small so that we keep just the first two terms of (45), then the temperature varies very approximately in a linear way with z .

In consequence, this observation is even more important, regarding that the proposed method is not restricted to small parameters by which increases the fruitfulness of the proposed method because it is possible to employ (45) instead of (49) for a wider interval of ε . Finally, maybe the main reason for proposing MHPMLT, is that the majority of non-linear differential equations that describe heat problems do not admit an exact solution.

Conclusions

In this work MHPMLT was employed to provide a handy analytical approximate solution for the second order non-linear differential equation which describes the steady-state 1-D heat conduction problem in a slab of thermal conductivity linearly dependent on the temperature defined with Dirichlet boundary conditions on a finite interval. The MHPMLT method expressed the problem of finding an approximate solution for the aforementioned differential equation, in terms of solving an algebraic equation for some unknown initial condition [8]. The square residual error of the approximate solutions shows that it is a method with high potential in the search for solutions of boundary value non-linear problems, even for the case of large perturbation parameters.

Finally, we note an important additional advantage from the proposed method; this does not require of solving several recurrence differential equations; indeed it requires only of calculating elementary LT in a systematic way. Therefore, LTHPM is a tool with high potential for practical applications in science and engineering [8].

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