THERMOSOLUTAL INSTABILITY IN A HORIZONTAL FLUID LAYER AFFECTED BY ROTATION

by

***Abdullah Ahmad ABDULLAH*[a]*, Sultana Dafer AL-AHMARI*[a]*, and Ali J. CHAMKHA*[b,c*]**

[a] Department of Mathematical Sciences, Faculty of Applied Science, Umm Al-Qura University, Makkah, Saudi Arabia
[b] Mechanical Engineering Department, Prince Sultan Endowment for Energy and Environment, Prince Mohammad Bin Fahd University, Al-Khobar, Saudi Arabia
[c] RAK Research and Innovation Center, American University of Ras Al Khaimah, Ras Al Khaimah, United Arab Emirates

Original scientific paper
https://doi.org/10.2298/TSCI171203093A

Thermosolutal convective instability in a horizontal layer affected by rotation is studied. Stationary convection and over-stability cases are considered for different boundary conditions. Analytical solutions were obtained when both boundaries are free and numerical results were obtained for the cases of free and rigid boundaries. The numerical computations of this problem were performed using the method of expansion of Chebyshev polynomials. This method is better suited to the solution of hydrodynamic stability problems than expansions in other sets of orthogonal polynomials. This method not only has high accuracy but also allows stationary and over-stable modes to be treated simultaneously, which is important if perchance the critical eigenvalue flits between the different modes in response to changing parameter values. The results obtained show that the effect of both solute concentration and rotation is to stabilize the system for stationary convection case and for the over-stability case when both boundaries are free. However, when both boundaries are rigid some unexpected behavior are obtained in the case of over-stability.

Key words: *thermal instability, thermosolutal convection, rotation, thermal Rayleigh number, concentration Rayleigh number*

Introduction

Thermal instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many fields of fluid dynamics. Rayleigh [1] provided a fundamental theoretical basis for the thermal instability in a fluid layer heated from below. The instability of a layer of fluid heated from below and subjected to Coriolis forces was first studied by Chandrasekhar [2] and Chandrasekhar and Elbert [3] for stationary convection and over-stability respectively. They showed that the effect of Coriolis forces on the instability of the fluid layer is to inhibit the onset of instability. Several other authors discussed the instability of fluids and the effect of rotation, Abdullah [4], Julien *et al.* [5], Chun *et al.* [6], Prosperetti [7], Geurts and Kunnen [8], Horn and Shishkina [9], Khan and Shafie [10], and Sharma *et al.* [11].

* Corresponding author, e-mail: achamkha@pmu.edu.sa

Previous studies, Stern [12], Walin [13], Veronis [14], and Nield [15], showed that the presence of salinity in convection problems usually has the effect of inhibiting the development of instabilities when the solute concentration decreases upwards and the fluid is heated from below. However, it has an opposite effect when the solute concentration increases upwards. Moreover, if the fluid is heated from above and soluted from below, then the temperature at the bottom of the fluid will be relatively high comparing to the other parts of the fluid. A comprehensive review of literature concerning thermosolutal convection can be found in Nield and Bejan [16]. Further studies in thermosolutal convection are presented in [17-24].

This work studies thermosolutal convective instability in a horizontal layer affected by rotation. Stationary convection and over-stability cases are considered for different boundary conditions. Analytical solutions were obtained when both boundaries are free and numerical results were presented for the cases of free and rigid boundaries. Many flows in nature are driven by buoyant convection and subsequently modulated by rotation. This problem can be expected to exist near coastal regions where tidal effects could carry salty warm water under fresher cold water, or in areas such as the eastern Atlantic where salty Mediterranean water flows out through the strait of Gibraltar into the open ocean. The problem is of great importance because of its application to atmospheric physics, oceanography, limnology and geophysical flows driven by rotation. Salt gradient solar ponds are a good application of this problem.

Formulation of the problem

Consider an infinite horizontal layer occupied by an incompressible, viscous, soluted fluid. The fluid is subject to a constant gravitational acceleration in the negative x_3 direction and is affected by rotation about the x_3 axis with a constant angular velocity Ω. The governing equations for this problem are:

$$v_{i,i} = 0 \tag{1}$$

$$\dot{v}_i = -\left(\frac{P}{\rho_0}\right)_{,i} + \nu\nabla^2 v_i - \mathrm{g}[1-\alpha(T-T_0)+\alpha'(c-c_0)]\delta_{i3} + 2\in_{ijk} v_j \Omega_k \tag{2}$$

$$\frac{\partial T}{\partial t} + v_j\frac{\partial T}{\partial x_j} = \kappa\frac{\partial^2 T}{\partial x_j^2} \tag{3}$$

$$\frac{\partial c}{\partial t} + v_j\frac{\partial c}{\partial x_j} = \kappa'\frac{\partial^2 c}{\partial x_j^2} \tag{4}$$

where v_i is the velocity components, P – the pressure, ν – the kinematic viscosity, g – the acceleration due to gravity, T – the temperature, c – the solute concentration, T_0, ρ_0, and c_0 are the temperature, density, and solute concentration at $x_3 = 0$, respectively, α – the coefficient of volume expansion, α' – the coefficient of solute expansion, κ – the coefficient of heat conduction, and κ' – the coefficient of solute diffusion. These equations are supplemented by an equation of state which depends on the Boussinesq approximation which state that density is constat everywhere except in the body force term in the eq. (2) of motion where the density is linearly proportional to temperature, T, and solute mass concentration, c, *i. e*:

$$\rho = \rho_0 \left[1 - \alpha(T - T_0) + \alpha'(T - T_0)\right] \tag{5}$$

The fluid is confined between the planes $x_3 = 0$ and $x_3 = d$ and on these planes, we need to specify mechanical, thermal and solute conditions.

The linearized equations

Equations (1)-(4) have a steady-state solution in which the fluid is at rest and the pressure, temperature and solute concentration are functions of x_3 alone. *i. e.*:

$$v_i = 0, \quad P = P(x_3), \quad T = T(x_3), \quad c = c(x_3) \tag{6}$$

Now, let the steady-state solution be slightly perturbed such that:

$$v_i = 0 + \epsilon\hat{v}_i, \quad T = T_0 - \beta x_3 + \epsilon\theta, \quad c = c_0 - \beta' x_3 + \epsilon\hat{c}, \quad P = P + \epsilon P$$

where $\hat{v}_i, \theta, \hat{c}$ and $\hat{P}$ are the linear perturbation of velocity, temperature, solute concentration and pressure, respectively. So the linearized form of eqs. (1)-(4) are:

$$\hat{v}_{i,i} = 0 \tag{7}$$

$$\frac{\partial \hat{v}_i}{\partial t} = -\left(\frac{\hat{P}}{\rho_0}\right)_{,i} + \nu\nabla^2\hat{v}_i + \mathrm{g}\left[\alpha\hat{\theta} - \alpha'\hat{c}\right]\delta_{i3} + 2\epsilon_{ijk}\hat{v}_j\Omega_k \tag{8}$$

$$\frac{\partial \hat{\theta}}{\partial t} - \beta\hat{v}_3 = \kappa\nabla^2\hat{\theta} \tag{9}$$

$$\frac{\partial \hat{c}}{\partial t} - \beta'\hat{v}_3 = \kappa'\nabla^2\hat{c} \tag{10}$$

Now, we use the non-dimensional variables x_i^*, t^*, v_i^*, c^*, θ^*, and P^* such that:

$$x_i = \mathrm{d}x_i^*, \quad t = \frac{\mathrm{d}^2}{\nu}t^*, \quad \hat{\upsilon}_i = \frac{\kappa}{d}\upsilon_i^*, \quad \hat{c} = \frac{\kappa}{d}\sqrt{\frac{\nu|\beta'|}{\kappa'\alpha' g}}c^*, \quad \hat{\theta} = \frac{\kappa}{d}\sqrt{\frac{\nu|\beta|}{\kappa\alpha g}}\theta^*, \quad \hat{P} = \frac{\rho_0\kappa\nu}{d^2}P^*$$

then eqs. (7)-(10) become:

$$v_{i,i} = o \tag{11}$$

$$\frac{\partial \upsilon_i}{\partial t} = -P_{,i} + \nabla^2 v_i + \sqrt{\mathrm{R_t}}\,\theta\delta_{i3} - \sqrt{\mathrm{R_s}}\,c\delta_{i3} + \sqrt{\mathrm{T_r}}\,\epsilon_{ijk}v_j\delta_{k3} \tag{12}$$

$$\mathrm{P_r}\frac{\partial \theta}{\partial t} + H\sqrt{\mathrm{R_t}}\,v_3 = \nabla^2\theta \tag{13}$$

$$\mathrm{P_r'}\frac{\partial c}{\partial t} + H'\sqrt{\mathrm{R_s}} = \nabla^2 c \tag{14}$$

where superscript "*" has been dropped but all the variables are now non-dimensional and where the non-dimensional numbers $\mathrm{R_t}$, $\mathrm{R_s}$, $\mathrm{P_r}$, $\mathrm{P_r'}$, and $\mathrm{T_r}$ are given by:

$$\mathrm{R_t} = \frac{\alpha g|\beta|}{\kappa\nu}d^4, \quad \mathrm{R_s} = \frac{g|\beta'|\alpha'}{\kappa'\nu}d^4, \quad \mathrm{P_r} = \frac{\nu}{\kappa}, \quad \mathrm{P_{r'}} = \frac{\nu}{\kappa'}, \quad \mathrm{T_r} = \frac{4\Omega^2}{\nu^2}d^4$$

where $\mathrm{R_t}$ is the thermal Rayleigh number, $\mathrm{R_s}$ – the concentration Rayleigh number, $\mathrm{P_r}$ – the viscous Prandtl number, $\mathrm{P_r}'$ – the Schmidt number, and $\mathrm{T_r}$ – the Taylor number and where:

$$H = -\frac{\beta}{|\beta|} = \begin{cases} 1 \text{when heating from above} \\ -1 \text{when heating from below} \end{cases}$$

$$H' = -\frac{\beta'}{|\beta'|} = \begin{cases} 1 \text{ when solute conetration increases upwards} \\ -1 \text{when solute conetration decreases upwards} \end{cases}$$

The boundary conditions

The fluid is confined between the planes $x_3 = 0$ and $x_3 = d$, and on these planes, we need to specify mechanical, thermal, and solutal conditions. Suitable mechanical conditions assume either a rigid boundary on which no slip occurs or a free boundary on which no tangential stresses act. Suitable thermal conditions assume either an insulating or a perfectly conducting boundary and suitable solutal conditions assume either a permeable or an impermeable boundary.

Mechanical conditions

For a rigid boundary, the no slip condition implies that the horizontal components of the fluid velocity and all of x_1, x_2 partial derivatives of each component of the fluid velocity vanish. Thus if $A_{,i}$ denotes the derivative of A with respect to i, then:

$$v_1 = v_2 = 0 \quad \text{and} \quad v_{1,1} = v_{2,2} = 0$$

and from the continuity eq. (11) we obtain $v_{3,3} = 0$. If we introduce ξ to be the fluid vorticity, then $\xi = \mathrm{curl}v$ and it is clear that $\xi_3 = 0$. For a free boundary no tangential stresses act, and we can show that $v_{3,33} = 0$, $\xi_{3,3} = 0$.

Thermal conditions

For a perfectly conducting boundary the temperatures of the boundary and impinging fluid match, whereas on a perfectly insulating boundary no heat transfer can take place between the fluid and the surrounding, hence the normal derivative of temperature is zero. In mathematical terms, the thermal conditions are:

$$\theta = \theta_{\mathrm{ext}}, \quad \text{on a conducting boundary}, \quad \theta_{,3} = 0, \quad \text{on an insulating boundary}$$

where θ_{ext} is the exterior non-dimensional temperature.

Solute conditions

The possible solute conditions are $c = c_{\mathrm{ext}}$, on a permeable boundary, $c_{,3} = 0$ on an impermeable boundary, where c_{ext} is the exterior non-dimensional concentration.

Normal mode analysis

In many convection problems, the vector components parallel to the direction of gravity (*i. e.* the x_3 direction) play a central role and so we introduce the variables w and ξ such that $w = v_3, \xi = \xi_3$. When we take the curl of eq. (12), we obtain:

$$\frac{\partial \xi_i}{\partial t} = \nabla^2 \xi_i + \sqrt{\mathrm{R_t}}\epsilon_{ijk}\frac{\partial \theta}{\partial x_j}\delta_{k3} - \sqrt{\mathrm{R_s}}\epsilon_{ijk}\frac{\partial c}{\partial x_j}\delta_{k3} + \sqrt{\mathrm{T_r}}\upsilon_{i,3} \tag{15}$$

Taking the curl again we obtain:

$$\frac{\partial}{\partial t}\nabla^2 \upsilon_i = \nabla^4 v_i - \sqrt{\mathrm{R_t}}(\theta_{,3i} - \nabla^2\theta\delta_{i3}) + \sqrt{\mathrm{R_s}}(c_{,3i} - \nabla^2 c\delta_{i3}) - \sqrt{\mathrm{T_r}}\xi_{i,3} \tag{16}$$

The third components of eqs. (13)-(16) are:

$$\frac{\partial \xi}{\partial t} = \nabla^2 \xi + \sqrt{\mathrm{T_r}} w_{,3} \tag{17}$$

$$\frac{\partial}{\partial t}\nabla^2 w = \nabla^4 w + \sqrt{\mathrm{R_t}}(\theta_{,11} + \theta_{,22}) - \sqrt{\mathrm{R_s}}(c_{,11} + c_{,22}) - \sqrt{\mathrm{T_r}}\xi_{,3} \tag{18}$$

$$\mathrm{P_r}\frac{\partial \theta}{\partial t} + H\sqrt{\mathrm{R_t}}w = \nabla^2\theta \tag{19}$$

$$\mathrm{P_r}'\frac{\partial c}{\partial t} + H'\sqrt{\mathrm{R_s}}w = \nabla^2 c \tag{20}$$

Now, we consider a solution of the form:

$$\varphi = \varphi(x_3)\exp\left[i(nx_1 + mx_2) + \sigma t\right]$$

where n, m are the wave numbers for harmonic disturbance and σ is the growth rate. Thus eqs. (17)-(20) become:

$$\sigma\xi = L\xi + \sqrt{T_r}Dw \tag{21}$$

$$\sigma Lw = L^2 w - a^2\sqrt{R_t}\theta + a^2\sqrt{R_s}C - \sqrt{T_r}D\xi \tag{22}$$

$$\sigma P_r\theta = L\theta - H\sqrt{R_t}w \tag{23}$$

$$\sigma P_r^{'}C = LC - H'\sqrt{R_s}\,w \tag{24}$$

where D is the operator $\partial/\partial x_3$, $a = \sqrt{(n^2 + m^2)}$ is the wave number, and $L = (D^2 - a^2)$. We may eliminate ξ, C and θ from eq. (22) to obtain:

$$\begin{aligned}&L^5 w + L^4 w[-\sigma P_r - 2\sigma - \sigma P_r^{'}] + L^3 w[2\sigma^2 P_r + \sigma^2 + \sigma^2 P_r^{'} P_r + 2\sigma^2 P_r^{'}] +\\ &+L^2 w[-\sigma^3 P_r - 2\sigma^3 P_r P_r^{'} - \sigma^3 P_r^{'} - a^2 R_t H + a^2 R_s H^{'}] +\\ &+Lw[\sigma^4 P_r P_r^{'} + \sigma a^2 R_t H - \sigma a^2 R_s H^{'} P_r - \sigma a^2 R_s H^{'}] + \sigma a^2 R_t P_r^{'} Hw - \sigma^2 a^2 R_t H P_r^{'} w +\\ &+\sigma^2 a^2 R_s H^{'} P_r w - \sigma T_r [P_r^{'} + P_r] L D^2 w + \sigma^2 T_r P_r P_r^{'} D^2 w + T_r L^2 D^2 w = 0\end{aligned} \tag{25}$$

which is a tenth order ODE to be satisfied by w. This problem has exact solutions only in the case of two free boundaries. However for the cases of two rigid boundaries or mixed boundaries it is not possible to obtain exact solutions and we have to rely on numerical methods. In the following section we discuss the analytical method when both boundaries are free, however in the result section we shall present numerical results for free and rigid boundaries.

The free boundary problem

In the following analysis, we shall consider both boundaries to be free. For the free boundary value problem $w = D^2w = 0$ on $x_3 = 0,1$. If we suppose that $w = \sin(l\pi x_3)$, is a suitable solution where A is a constant and l is an integer, then $Dw = Al\pi\cos l\pi x_3$, $D^2w = -Al^2\pi^2\sin l\pi x_3$, $Lw = \lambda w$ were $\lambda w = l^2\pi^2 + a^2$ and (25) becomes

$$\begin{gathered}\sigma^4 P_r P_r' + \sigma^3(P_r + P_r' + 2P_r P_r')\lambda + \sigma^2\{[1 + P_r P_r' + 2(P_r + P_r')\lambda^2] + \\ +(a^2 R_t H P_r' - a^2 R_s H' P_r + T_r P_r P_r' l^2\pi^2)\lambda^{-1}\} + \sigma[(2 + P_r + P_r')\lambda^3 + \\ +a^2 R_t H(1 + P_r') - a^2 R_s H'(1 + P_r) + T_r l^2\pi^2(P_r + P_r')] + \lambda^4 + \\ +a^2 R_t H\lambda - a^2 R_s H'\lambda + T_r l^2\pi^2\lambda = 0\end{gathered} \tag{26}$$

The solutions of eq. (26) are functions of P_r, P'_r, T_r, R_s, and R_t and we have to examine how the nature of these solutions depends on these variables by considering the following cases. Note that because of the complexity of eq. (26) it is not possible to discuss analytically the effect of over-stability but later on we shall discuss its effect from the numerical results.

Case (1): when the fluid is heated from above and the solute concentration decreases upwards

Here $H = 1$, $H' = -1$. So we need to discuss the roots of the polynomial equation $f(\sigma) = 0$ where

$$\begin{gathered}f(\sigma) = \sigma^4 P_r P_r' + \sigma^3(P_r + P_r' + 2P_r P_r')\lambda + \\ +\sigma^2\{[1 + P_r P_r' + 2(P_r + P_r')]\lambda^2 + (a^2 R_t P_r' - a^2 R_s P_r + T_r P_r P_r' l^2\pi^2)\lambda^{-1}\} + \\ +\sigma[(2 + P_r + P_r')\lambda^3 + a^2 R_t(1 + P_r') - a^2 R_s(1 + P_r) + T_r l^2\pi^2(P_r P_r')] + \lambda^4 + \\ +a^2 R_t\lambda + a^2 R_s\lambda + T_r l^2\pi^2\lambda = 0\end{gathered} \tag{27}$$

To obtain the critical thermal Rayleigh number for the onset of stationary convection case we set $\sigma = 0$ in eq. (27). Thus

$$R_t = -\left[\frac{\lambda^3}{a^2} + \frac{T_r l^2\pi^2}{a^2} + R_s\right], \quad l^2 = 1 \tag{28}$$

Since R_t is negative then no stationary convection happened.

Case (2): when the fluid is heated from above and the solute concentration increases upwards

Here $H = 1$, $H' = -1$. To obtain the critical thermal Rayleigh number for the onset of stationary convection case we set $\sigma = 0$ in eq. (27). Thus

$$R_t = -\left[\frac{\lambda^3}{a^2} + \frac{T_r l^2\pi^2}{a^2} - R_s\right] \tag{29}$$

Clearly stationary stability is possible provided $R_s > 1/a^2(\lambda^3 + T_r l^2\pi^2)$. Moreover:

$$\frac{\mathrm{d}R_t}{\mathrm{d}R_s}=1, \quad \frac{\mathrm{d}R_t}{\mathrm{d}T_r}=-\frac{l^2\pi^2}{a^2} \tag{30}$$

So the solute concentration has a stabilizing effect on the system, but the rotation has a destabilizing effect on the system in this case.

Case (3): when the fluid is heated from below and the solute concentration increases upwards

Here $H = 1$, $H' = -1$. So from eq. (27) the thermal Rayleigh number for the onest of stationary convection has the form:

$$R_t=\frac{\lambda^3}{a^2}+T_r\frac{l^2\pi^2}{a^2}-R_s \tag{31}$$

Clearly stationary stability is possible provided $\lambda^3/a^2 + T_r(l^2\pi^2)/a^2 > R_s$. Moreover:

$$\frac{\mathrm{d}R_t}{\mathrm{d}R_s}=-1, \quad \frac{\mathrm{d}R_t}{\mathrm{d}T_r}=\frac{l^2\pi^2}{a^2} \tag{32}$$

So it is clear in this case that the solute concentration has a destabilizing effect on the system but the rotation has a stabilizing effect on the system.

Case (4): when the fluid is heated from below and the solute concentration decreases upwards

Here $H = 1$, $H' = -1$. So from eq. (27) the thermal Rayleigh number for the onset of stationary convection has the form $R_t = \lambda^3/a^2 + R_s + l^2\pi^2(T_r/a^2)$. From which we can see that:

$$\frac{\mathrm{d}R_t}{\mathrm{d}R_s}=1, \quad \frac{\mathrm{d}R_t}{\mathrm{d}T_r}=\frac{l^2\pi^2}{a^2} \tag{33}$$

It is clear that the solute concentration and rotation have a stabilizing effect on the system.

Numerical method

The numerical computations of this problem are performed using the method of expansion of Chebyshev polynomials. This method is better suited to the solution of hydrodynamic stability problems than expansions in other sets of orthogonal polynomials. Chebyshev method has been used to obtain numerical solutions of thermal stability problems by several authors (Orszag [25], Hassanien and El-Hawary [26], Abdullah and Lindsey [27], Hassanien *et al.* [28], Straughan [29], Banjer and Abdullah [30]). The eigenvalue problem of eqs. (21)-(24) together with the boundary conditions are to be solved numerically for the case when the fluid layer is heated from below and the solute concentration decreases upwards which corresponds to case of eq. (4) in the previous section. This eigenvalue problem can be solved using free or rigid boundaries for the two cases of solute conditions. For free boundaries the conditions are $w=\theta=\varnothing=D\xi=0, x_3=0,1$ and for rigid boundaries the conditions are $w=\theta=Dw=\xi=0,\ x_3=0,1$. The interval (0, 1) is first mapped into the interval (–1, 1) by the transformation $z=2x_3-1$. Thereafter, the variables of the problem are assigned the Chebyshev spectral expansion:

$$y_r(z) = \sum_{k=0}^{N-1} a_{rk} T_k(z) \tag{34}$$

where N is a user-specified number of Chebyshev polynomials. The system of ODE (21)-(24) and boundary conditions lead to a generalized eigenvalue problem of type $AY = \sigma BY$ in which B is a singular matrix. The eigenvalues, σ, and corresponding eigenvectors are calculated using a specialized routine.

Results and discussion

Both boundaries are free

The relation between the concentration Rayleigh number, R_s, and the critical thermal Rayleigh number, R_{ct}, for the stationary convection case is displayed in fig. 1 for different values of the Taylor number, T_r. It is clear from this figure that as R_s increases, R_{ct} increases which indicates that the solute concentration has a stabilizing effect on the system. Moreover the figure shows that as T_r increases R_{ct} increases which indicates that rotation has a stabilizing effect on the system. These results coincide with the results obtained in eq. (33) in the analytical solution of the problem.

In the case of over-stability, the relation between the concentration Rayleigh number, R_s, and the critical thermal Rayleigh number, R_{ct} for different values of the Taylor number, T_r, is displayed in fig. 2 when $P_r = 0,5$, $P'_r = 0,025$. It is clear from this figure that as R_s increases R_{ct} increases which indicates that the solute concentration has a stabilizing effect on the system for the over-stable case also. Moreover the figure shows that as T_r increases R_{ct} increases which indicates that rotation has a stabilizing effect on the system. The numerical results in this case show that there is a condition for over-stability to ensue. This condition depends on the values of the Taylor number, T_r and the concentration Rayleigh number, R_s, As T_r decreases over-stability is possible provided that R_s exceeds a certain value as shown in fig. 2. Figure 3 shows a comparison between stationary convection and over-stability cases. In this figure a relation between the concentration Rayleigh number, R_s, and the critical thermal Rayleigh number, R_{ct}, is displayed when $T_r = 0.10^5$. When $T_r = 0$, stationary convection is the preferred mechanism if $R_s < 500$ otherwise over-stability is the preferred mechanism. When $T_r = 10^5$ over-stabilty is possible provided $R_s \geq 100$ and if this condition is satisfied then over-stability is always the preferred mechanism.

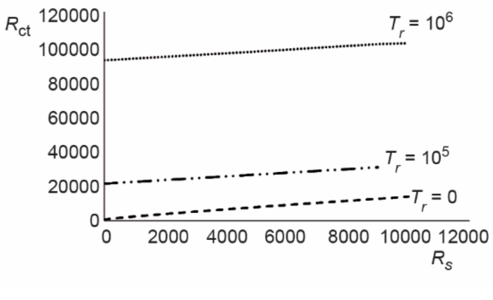

Figure 1. The relation between R_s and R_{ct} for the stationary convection case when both boundaries are free for different values of T_r

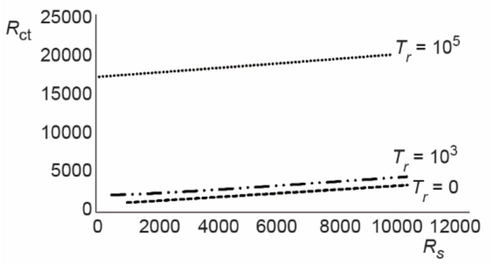

Figure 2. The relation between R_s and R_{ct} for the over-stability case when both boundaries are free for different values of T_r when $P_r = 0,5$

Both boundaries are rigid

For the stationary convection case, the effect of the concentration Rayleigh number, R_s, on the critical Rayleigh number, R_{ct}, is similar to that of the free boundary case as shown in fig. 4. However the effect of the Taylor number, T_r, in this case is unexpected. The figure shows that as T_r increases the critical Rayleigh number, R_{ct}, decreases which indicates that rotation has a destabilizing effect in this case. This behavior appears when $R_s > 500$ but below this value we notice that as T_r increases the critical Rayleigh number, R_{ct}, increases. In the case of over-stability, the effects of the concentration Rayleigh number, R_s, and the Taylor number, T_r, on the critical thermal Rayleigh number, R_{ct}, are displayed in fig. 5. These effects are similar to those of the stationary convection case. A comparison between stationary convection and over-stability cases is displayed in fig. 6. In this figure, a relation between the concentration Rayleigh number, R_s, and the critical thermal Rayleigh number, R_{ct}, is displayed when $T_r = 0$. Here the preferred mechanism flits between stationary convection case and over-stability case.

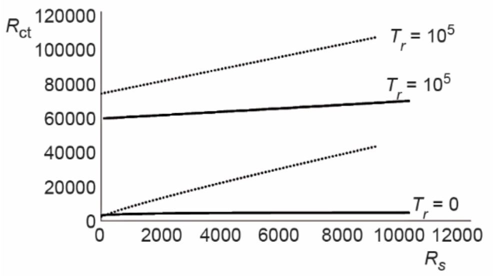

Figure 3. A comparison between stationary convection and over-stability cases when both boundaries are free for different values of T_r

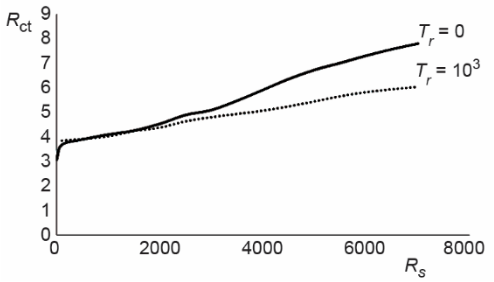

Figure 4. The relation between R_s and R_{ct} for the stationary convection case when both boundaries are rigid for different values of T_r

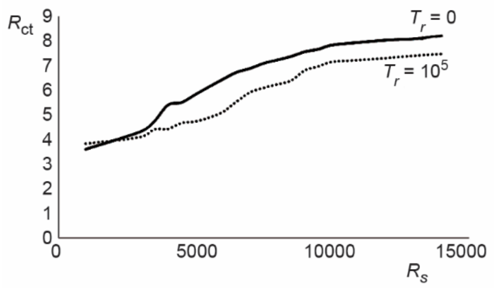

Figure 5. The relation between R_s and R_{ct} for the over-stability case when both boundaries are rigid for different values of T_r

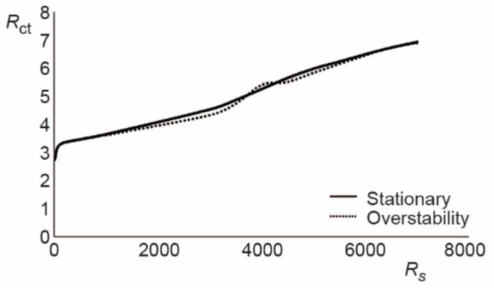

Figure 6. A comparison between stationary convection and over-stability cases when both boundaries are rigid. Here $T_r = 0$

Conclusion

This work studies thermosolutal convective instability in a horizontal layer affected by rotation. Analytical solutions are obtained for the free boundary problem in the case of stationary convection. Typical cases are discussed analytically depending on heating and solute concentration in both boundaries. Numerical results are obtained using the method of expansion of Chebyshev polynomials for the case when the layer of fluid is heated from below and the solute concentration decreases upwards when both boundaries ar′e free and when both boundaries are rigid. The results obtained for the free boundary case coincide with the analytical solutions obtained. The effect of both solute concentration and rotation is discussed and results show that their effect is to stabilize the system for stationary convection case and for the over-stability case when both boundaries are free. However when both boundaries are rigid some unexpected behavior are obtained in the case of over-stability.

Acknowledgment

The authors would like to thank Deanship of Scientific Research at Umm Al-Qura University (Project ID 43405021) for the financial support.

Nomenclature

a – wave number [cm^{-1}]
c – solute concentration [kgm^{-3}]
c_{ext} – exterior solute concentration [kgm^{-3}]
d – thickness of fluid layer, [m]
g – gravitational acceleration, [ms^{-2}]
P – pressure, [Nm^{-2}]
$\mathrm{P'_r}$ – Schmidt number, $(= \nu/k')$
$\mathrm{P_r}$ – viscous Prandtle number, $(= \nu/k)$
$\mathrm{R_s}$ – concentration Rayleigh number, $\{=[(g|\beta'|\alpha')/k'\nu]d^4\}$
$\mathrm{R_t}$ – thermal Rayleigh number, $\{=[(\alpha g|\beta|)/k\nu]d^4\}$
T – temperature [°C]
t – time, [s]
$\mathrm{T_r}$ – Taylor number, $\{=[(4\Omega^2)/\nu^2]d^4\}$
$\boldsymbol{v}$ – velocity, [ms^{-1}]

Superscripts

x_i – co-ordinates, (i =1, 2, 3)
^ – linear perturbation quantity

Greek symbols

α – coefficient of volume expansion, [K^{-1}]
α' – coefficient of solute expansion, [kgm^{-3}]
β – adverse temperature gradient, [Km^{-1}]
β' – adverse concentration gradient [kgm^{-4}]
δ_{ij} – Kronecker delta
ϵ_{ijk} – permutation tensor
θ – dimensionless temperature
θ_{ext} – exterior dimensionless temperature
κ – heat conduction coefficient, [Wm^{-1}K^{-1}]
κ' – solute diffusion coefficient, [m^{2}s^{-1}]
ν – kinematic viscosity, [kg/m^{-1}s^{-1}]
$\boldsymbol{\xi}$ – fluid vorticity, [s^{-1}]
ρ – density of fluid layer, [kgm^{-3}]
σ – growth rate
Ω – angular velocity, [rads^{-1}]

Subscript

o – referential quantity

References

[1] Rayleigh, L., On Convection Currents in a Horizontal Layer of Fluid when the Higher Temperature is on the Underside, *Philosophical Magazine, 32* (1916), 192, pp. 529-546
[2] Chandrasekhar, S. *Hydrodynamic and Hydromagnetic stability*, Dover Publications Inc. New York, USA 1961
[3] Chandrasekhar, S., Elbert, D. The Instability of a Layer of Fluid Heated Below and Subject to Coriolis Forces. II. *Proceeding of the Royal Society of London A, 231* (1955), pp. 198-210
[4] Abdullah, A. Thermal Instability of a Non-Linear Magnetic Fluid Under the Influence of both Non-Vertical Magnetic Field and Coriolis Forces, *Arabian Journal of Science and Engineering, 17* (1992), 4B, pp. 625-633
[5] Julien, K., *et al.*, Rapidly Rotating Turbulent Rayleigh-Benard Convection, *Journal of Fluid Mechanics, 322* (1996), Sept., pp. 243-273
[6] Chun, H., *et al.*, Stratified Rotating Boussinesq Equations in Geophysical Fluid Dynamics: Dynamic Bifurcation and Periodic Solutions, *Journal of Mathematical Physics, 48* (2007), 6, 065602
[7] Prosperetti, A., The Effect of Rotation on the Rayleigh-Bénard Stability Threshold, *Physics of Fluids, 24* (2012), 11, pp. 114101-114116
[8] Geurts, B., Kunnen, R., Intensified Heat Transfer in Modulated Rotating Rayleigh-Bénard Convection, *International Journal of Heat and Fluid Flow, 49* (2014), Oct., pp. 62-68
[9] Horn, S., Shishkina, O., Toroidal and Poloidal Energy in Rotating Rayleigh-Bénard Convection , *Journal of Fluid Mechanics, 762* (2015), Jan., pp. 232-255

[10] Khan, I., Shafic, S., Rotating MHD Flow of A Generalized Burgers' Fluid Over an Oscillating Plate Embedded in a Porous Medium, *Thermal Science, 19* (2015), Suppl.1, pp. S183-S190
[11] Sharma, P. *et al.*, Rayleigh Taylor Instability in Dusty Magnetized Fluids with Surface Tension Flowing Through Porous Medium, *Thermal Science, 20* (2016), 1, pp. 119-130
[12] Stern, M., The Salt Fountain and Thermohaline Convection, *Tellus,12* (1960), 2, pp. 172-175
[13] Walin, G., Note on the Stability of Water Stratified by both Salt and Heat, *Tellus, 16* (1964), 3, pp. 389-393
[14] Veronis, G., On Finite Amplitude Instability in Thermohaline Convection, *Journal of Marine Research, 23* (1965), 1, pp. 1-17
[15] Nield, D., The Thermohaline Rayleigh-Jeffreys Problem, *Journal of Fluid Mechanics, 29* (1967), 3, pp. 545-558
[16] Nield, D., Bejan, A., *Convection in Porous Media*, Springer, New York, USA, 2006
[17] Murthy, J., Lee, P., Thermosolutal Convection in a Floating Zone: The Case of an Unstable Solute Gradient, *International Journal of Heat and Mass Transfer, 31* (1988), 9, pp. 1923-1932
[18] Boudourides, M., Nikoudes, A., The Attractor of Thermosolutal Convection, *Physica D, 47* (1991), 3, pp. 439-449
[19] Abdullah, A., Thermosolutal Convection in a Non-Linear Magnetic Fluid. *International Journal of Thermal Sciences, 39* (2000), 2, pp. 273-284
[20] Xu, L., Yang, S., Stability Analysis of Thermosolutal Second-Order Fluid in Porous Bénard Layer. *Ricerche di Matematica, 56* (2007), 1, pp. 149-160
[21] Jena, S., *et al.*, Thermosolutal Convection in a Rectangular Concentric Annulus: A Comprehensive Study**,** *Transport in Porous Media, 98* (2013), 1, pp. 103-124
[22] Sankar, M., *et al.*, Thermosolutal Convection from a Discrete Heat and Solute Source in a Vertical Porous Annulus, *International Journal of Heat and Mass Transfer, 55* (2012), 15-16, pp. 4116-4128
[23] Costa, V., Double-Diffusive Natural Convection in Parallelogrammic Enclosures Filled with Fluid Saturated Porous Media, *International Journal of Heat and Mass Transfer, 47* (2004), 12-13, pp. 2699- 2714
[24] Jagadeesha, R., *et al.*, Numerical Simulation of Double Diffusive Magnetoconvection in an Inclined Parallelogrammic Porous Enclosure with an Internal Heat Source, *Materials Today: Proceedings, 4* (2017), 9, pp. 10544-10548
[25] Orzsag, S., Accurate Solution of the Orr-Sommerfeld Equation, *Journal of Fluid Mechanics, 50* (1971), 4, pp. 689-703
[26] Hassanien, I., El-Hawary, H., Chebyshev Solution of Laminar Boundary Layer Flow, *International Journal of Computer Mathematics, 33* (1989), 1-2, pp. 127-132
[27] Abdullah, A., Lindsay, k., Benard Convection in a Non-Linear Magnetic Fluid, *Acta Mechanica, 85* (1990), 1-2, pp. 27-42
[28] Hassanien, I., *et al.*, Chebyshev Solution of Axisymmetric Stagnation Flow on a Cylinder, *Energy Conversion and Management, 37* (1996), 1, pp. 67-76
[29] Straughan, B., Effect of Property Variation and Mode on Convection in a Fluid Overlying a Porous layer, *International Journal for Numerical and Analytical Methods in Geomechanics, 26* (2002), 1, pp. 75-97
[30] Banjer, H., Abdullah, A., Thermal Instability in Superposed Porous and Fluid Layers in the Presence of Coriolis Forces, *International Journal of Applied Mathematics and Mechanics, 7* (2011), Jan., pp. 13-27

Paper submitted: December 3, 2017
Paper revised: February 20, 2018
Paper accepted: February 21, 2018