INVARIANT APPROACHES FOR THE ANALYTIC SOLUTION OF THE STOCHASTIC BLACK-DERMAN TOY MODEL

by

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We work on the analytical solution of the stochastic differential equations (SDE) via invariant approaches. In particularly, we focus on the stochastic Black-Derman Toy (BDT) interest rate model, among others. After we present corresponding (1+1) parabolic linear PDE for BDT-SDE, we use theoretical framework about the invariant approaches for the (1+1) linear PDE being done in the literature. We show that it is not possible to reduce BDT-PDE into the first and second Lie canonical forms. On the other hand, we success to find transformations for reducing it to the third Lie canonical form. After that, we obtain analytical solution of BDT-PDE by using these transformations. Moreover, we conclude that it can be reduced to the fourth Lie canonical form but, to the best of our knowledge, its analytical solution in this form is hard to find yet.

Key words: *heat equations, canonical Lie forms, invariant approaches, Black-Derman Toy model, analytical solution, stochastic model*

Introduction

Stochastic models are generally used by the investors in the financial markets. Most of these models do not have analytical solution so that they have to use its numerical approximation, which is obtained by performing the simulations or using any other numerical methods, in their analysis. In this article, we work on analytical solution of the stochastic models in the light of the invariant approaches and focus on BDT interest rate model, among others. While we explore the existence of its analytical solution by the Lie symmetry sense, we firstly show that it can not be written in terms of the first and second Lie canonical forms. Later, we are able to find transformations, which reduce it to the third Lie canonical form, and obtain analytical solution of BDT-PDE. We actually believe that its solution may help the practitioners who are willing to use this model in their works. Although they can quantify and analyze their investigation with this model using stochastic calculus tools, the analytical solution of it may help them to obtain relatively more sensitive analysis results which are the most important part for the investors throughout the decision process.

In the asset price process the interest rate is the one of the crucial parameter and it can be changed randomly in the real market. Therefore, risk takers want to minimize their risk using the most suitable stochastic model for their problem and obtain realistic solutions with respect to its analytic or numerical solution. As we mentioned before the stochastic models, which are

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described with SDE or system of differential equations, do not have analytic solution in general. In this case, the usage of the invariant approaches based on the Lie symmetry analysis is increased by the most of the scientist in the last decades. For instance, Lie symmetry analysis of the Merton-Black Scholes [1] model is one of the first work is done in the literature [2]. Moreover, Bakkaloglu *et al.* [3] examine the invariant approach to optimal investment consumption problem in. Furthermore, Pooe *et al.* in [4] derive the fundamental solutions to the zero-coupon bond pricing equations and Mahomed *et al.* [5] study invariant approaches to equations of finance in. On the other hand, Izgi and Bakkaloglu˙ [6, 7] investigate the deterministic solutions of the some stochastic differential equations. They work on the Ho-Lee stochastic interest rate model and obtain the fundamental solution of this model in terms of the heat equation solution and they also show some important results for the calibration of the model parameters via simulations [7].

Black-Derman Toy stochastic interest rate model

The BDT model is one of the first no-arbitrage interest rate model which was introduced by Black, Derman and Toy [8]. It is a special case of the more general Black-Karasinski model [9]. It is considerably consistent with the term structure of interest rates which is observed in the financial market. Bond pricing, option pricing and modeling future interest rate are some of the applications area of this model in the literature.

Its discrete-time version, which is quite famous, can be constructed in the form of a binomial tree. On the other hand, it also has continuous-time version which represents some theoretical difficulties. We work on its continuous-time version to find analytic solution of BDT model using invariant criteria to overcome these difficulties.

Its SDE, where the continuous-time version of BDT interest rate model follows a normal process:

$$dr(t) = r(t)\left[d(t)dt + \sigma dW(t)\right] \tag{1}$$

Here, the drift term $d(t)$ is a deterministic function of time which is the main differences between BDT and Black Scholes models. In the financial applications with such model, whose drift term is deterministic function, drift term should be determined carefully otherwise the interest rate can be reached to the negative values. The constant diffusion term, σ, represents volatility parameter of interest rate process while $r(t)$ and $W(t)$ represent the interest rate and 1-D standard Brownian motion, respectively.

Moreover, BDT derived this no-arbitrage interest rate model (see [8]) with respect to the complete market assumptions (*i. e.* no taxes, no transaction costs, no arbitrage opportunity, *etc.*) so that the zero coupon price $u(x,t)$ in the BDT model satisfies the scalar linear (1 + 1) parabolic PDE:

$$u_t = -\frac{1}{2}\sigma^2 x^2 u_{xx} - d(t)xu_x + xu, \;\; u(x,T) = 1 \tag{2}$$

This equation is referred as linear (1 + 1) parabolic BDT-PDE.

Review of theoretical framework: invariant criteria for linear (1 + 1) parabolic PDE

In this section, we briefly introduce the main results of Mahomed on the invariant characterization of scalar linear (1 + 1) parabolic PDE (for more details see [10]).

The general representation of the scalar linear (1+1) parabolic PDE of one time and one space variable is:

$$\frac{\partial u}{\partial t}=a\left(t,x\right)\frac{\partial^2 u}{\partial x^2}+b\left(t,x\right)\frac{\partial u}{\partial x}+c\left(t,x\right)u,\ (3) \tag{3}$$

where the coefficients *a, b,* and *c* are the continuous functions of *t* and *x*.

Lie [11] showed the relationship between the scalar linear parabolic PDE (3) and four Lie canonical forms. In this point, the following theorems (see [10]) which provide invariant criteria for the reduction of scalar linear (1+1) parabolic PDE (3) into different Lie canonical forms, take important role.

Theorem 1. [10] The linear parabolic eq. (3) is reducible to the classical heat PDE (or the first Lie canonical form)

$$\frac{\partial \overline{u}}{\partial \overline{t}}=\frac{\partial^2 \overline{u}}{\partial \overline{x}^2}$$

via the transformations:

$$\overline{t}=\phi(t)$$

$$\overline{x}=\pm\int\left[\dot{\phi}a(t,x)^{-1}\right]^{1/2}\mathrm{d}x+\beta(t)$$

$$\overline{u}=\upsilon(t)\left[a(t,x)\right]^{-1/4}u\exp\left\{\int\frac{b(t,x)}{2a(t,x)}\mathrm{d}x-\frac{1}{8}\frac{\ddot{\phi}}{\dot{\phi}}\left[\int\frac{dx}{a(t,x)^{1/2}}\right]^2\right\}-$$

$$-\frac{1}{2}\int\frac{1}{a(t,x)}\frac{\partial}{\partial t}\left[\int\frac{\mathrm{d}x}{a(t,x)^{1/2}}\right]\mathrm{d}x\pm\frac{1}{2}\frac{\dot{\beta}}{\dot{\phi}^{1/2}}\int\frac{\mathrm{d}x}{a(t,x)^{1/2}} \tag{4}$$

where $\dot{\phi}$ and a have the same sign, and ϕ,β, and υ satisfy:

$$\dot{\phi}\overline{c}=J+\frac{\partial}{\partial t}\int\frac{b(t,x)}{2a(t,x)}\mathrm{d}x-\frac{1}{2}\int\frac{1}{a(t,x)^{1/2}}\frac{\partial^2}{\partial t^2}\left[\int\frac{\mathrm{d}x}{a(t,x)^{1/2}}\right]\mathrm{d}x+$$

$$+f(t)\left[\int\frac{\mathrm{d}x}{a(t,x)^{1/2}}\right]^2+g(t)\left[\int\frac{\mathrm{d}x}{a(t,x)^{1/2}}\right]+h(t) \tag{5}$$

with *J* is:

$$J=c-\frac{b_x}{2}+\frac{ba_x}{2a}+\frac{a_{xx}}{4}-\frac{3}{16}\frac{{a_x}^2}{a}-\frac{a_t}{2a}-\frac{b^2}{4a} \tag{6}$$

and

$$f(t)=\frac{1}{16}\frac{(\ddot{\phi})^2}{(\dot{\phi})^2}-\frac{1}{8}\left(\frac{\ddot{\phi}}{\dot{\phi}}\right)_t$$

$$g\left(t\right)=\pm\frac{1}{8}\frac{\ddot{\phi}}{\dot{\phi}}\frac{\dot{\beta}}{(\dot{\phi})^{1/2}}\pm\frac{1}{2}\left[\frac{\dot{\beta}}{(\dot{\phi})^{1/2}}\right]_t$$

$$h\left(t\right)=\frac{1}{4}\frac{\ddot{\phi}}{\dot{\phi}}+\frac{1}{4}\frac{(\dot{\beta})^2}{\dot{\phi}}+\frac{\dot{\vartheta}}{\vartheta} \tag{7}$$

The functions *f, g,* and *h* are constrained by the relation defined in eq. (5) (see [10] and references therein).

Theorem 2. [10] The following are equivalent statements:

(a) the scalar linear (1+1) parabolic PDE (3) has six non-trivial point symmetries in addition to the infinite number of superposition symmetries,

(b) the coefficients of parabolic eq. (3) satisfies the invariant equation:

$$2L_x + 2M_x - 2N_x, = 0 \tag{8}$$

where

$$L = \sqrt{a}\left[\sqrt{a}J_x\right]_x, \quad M = \sqrt{a}\left[\sqrt{a}\partial_t\left(\frac{b}{2a}\right)\right]_x, \quad N = \sqrt{a}\partial_t^2\left(\frac{1}{\sqrt{a}}\right) \tag{9}$$

and J is given by eq. (6),

(c) the linear parabolic eq. (3) is reducible to the classical heat PDE $\partial\overline{u}/\partial\overline{t} = \partial^2\overline{u}/\partial\overline{x}^2$ via the transformations (4) for which $\varphi, \beta,$ and v are constructed from eq. (7) with the functions *f, g,* and *h* are constrained by the relation:

$$J + \partial_t \int \frac{b}{2a}\mathrm{d}x - \frac{1}{2}\int\frac{1}{\sqrt{a}}\partial_t^2\left(\int\frac{\mathrm{d}x}{\sqrt{a}}\right)\mathrm{d}x + f(t)\left(\int\frac{\mathrm{d}x}{\sqrt{a}}\right)^2 + g(t)\int\frac{\mathrm{d}x}{\sqrt{a}} + h(t) = 0 \tag{10}$$

Mahomed [10] stated necessary and sufficient conditions for reduction (1+1) parabolic PDE (3) into the second, third or fourth Lie canonical form by his following theorems:

Theorem 3. [10] If the scalar linear (1+1) parabolic PDE (3) does not satisfy condition (8) then it can be reduced to the second Lie canonical form:

$$\frac{\partial\overline{u}}{\partial\overline{t}} = \frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \frac{A}{\overline{x}^2}, \quad A \neq 0$$

is a constant when the following necessary and sufficient conditions hold:

$$\begin{aligned}
&20L_x + 20M_x - 10N_x + 10\left[\sqrt{|a|}M_x\right]_x \int\frac{\mathrm{d}x}{\sqrt{|a|}} - 5\left[\sqrt{|a|}N_x\right]_x\int\frac{\mathrm{d}x}{\sqrt{|a|}} + \\
&+10\left[\sqrt{|a|}L_x\right]_x\int\frac{\mathrm{d}x}{\sqrt{|a|}} + \left[\sqrt{|a|}\left[\sqrt{|a|}L_x\right]_x\right]_x\left(\int\frac{\mathrm{d}x}{\sqrt{|a|}}\right)^2 + \\
&+\left[\sqrt{|a|}\left[\sqrt{|a|}M_x\right]_x\right]_x\left(\int\frac{\mathrm{d}x}{\sqrt{|a|}}\right)^2 - \frac{1}{2}\left[\sqrt{|a|}\left[\sqrt{|a|}N_x\right]_x\right]_x\left(\int\frac{\mathrm{d}x}{\sqrt{|a|}}\right)^2 = 0
\end{aligned} \tag{11}$$

where *L, M, N,* and *J* are defined in eqs. (9) and (6).

Theorem 4. [10] If the scalar linear (1+1) parabolic eq. (3) does not satisfy condition (8) in *Theorems 2* and condition (11) in *Theorem 3* then it is equivalent to the third Lie canonical form:

$$\frac{\partial\overline{u}}{\partial\overline{t}} = \frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \overline{c}\left(\overline{x}\right)\overline{u}$$

if and only if the coefficients of parabolic eq. (3) satisfy the invariant criterion:

$$\frac{\partial}{\partial t}\left[J+\frac{\partial}{\partial t}\int\frac{b}{2a}\mathrm{d}x-\frac{1}{2}\int\frac{1}{\sqrt{|a|}}\left(\frac{\partial^2}{\partial t^2}\int\frac{\mathrm{d}x}{\sqrt{|a|}}\right)\mathrm{d}x\right]=0 \tag{12}$$

via the transformations:

$$\overline{t}=\epsilon t+a_1,\quad \epsilon=\pm 1,\quad a_1=\text{const.}$$

$$\overline{x}=\pm\int\left[\epsilon a(t,x)\right]^{-0.5}\mathrm{d}x$$

$$\overline{u}=\upsilon_0\left|a(t,x)\right|^{-0.25}u\exp\left\{\int\frac{b(t,x)}{2a(t,x)}\mathrm{d}x-\frac{1}{2}\int\frac{1}{a(t,x)^{0.5}}\partial_t\left[\int\frac{\mathrm{d}x}{a(t,x)^{0.5}}\right]\mathrm{d}x\right\} \tag{13}$$

here υ_0 is a constant. If $a>0$, then $\epsilon=1$, otherwise $\epsilon=-1$. The $\overline{c}$ in the transformed PDE should satisfy:

$$\varepsilon\overline{c}=J+\frac{\partial}{\partial t}\int\frac{b}{2a}\mathrm{d}x-\frac{1}{2}\int\frac{1}{\sqrt{|a|}}\left(\frac{\partial^2}{\partial t^2}\int\frac{\mathrm{d}x}{\sqrt{|a|}}\right)\mathrm{d}x$$

Theorem 5. [10] If the scalar linear (1+1) parabolic eq. (3) does not satisfy the conditions of *Theorem 2-4* then it is reducible to the fourth Lie canonical form:

$$\frac{\partial\overline{u}}{\partial\overline{t}}=\frac{\partial^2\overline{u}}{\partial\overline{x}^2}+\overline{c}\left(\overline{x},\overline{t}\right)\overline{u} \tag{14}$$

Analytic solution of BDT linear (1 + 1) parabolic PDE

In this section, we present invariant approach to analytic solution of the BDT model. We investigate the possible transformations of BDT-PDE in eq. (2) for the related Lie canonical forms under the consideration of the theorems are given in section *Review of theoretical framework: invariant criteria for linear* (1 + 1) *parabolic PDE*.

Transformation: BDT-PDE to 1st Lie canonical form

The coefficients of the scalar linear (1+1) parabolic BDT-PDE (2) are as follows:

$$a(t,x)=-\frac{1}{2}\sigma^2x^2,\quad b(t,x)=-d(t)x,\quad c\left(t,x\right)=x \tag{15}$$

where the $a(t,x)$, $b(x,t)$, and $c(x,t)$ are defined in eq. (3). Now, we need to check whether BDT-PDE satisfies *Theorem 2* conditions or not. First, we evaluate *J* as following which is given by eq. (6):

$$J=x\frac{1}{2}d(t)+\frac{1}{2}\frac{d(t)^2}{\sigma^2}+\frac{\sigma^2}{8} \tag{16}$$

We can easily obtain $L=(1/2)\sigma^2x$ using this *J* in eq. (9). Moreover, we see that *M* and *N* are 0 when we evaluate them by using eq. (9) with respect to the coefficients in eq. (15). If we substitute these values in the invariant condition (8) in *Theorem 2*, then $2L_x+2M_x-2N_x=0$

reduces to the $\sigma^2 = 0$. This is not the case which we are interested in since the BDT model's SDE will become ODE for $\sigma = 0$, and also the BDT-PDE will not be parabolic with this value. As a result, σ can not take zero value and *Theorem 2* conditions failed to hold. Furthermore, BDT-PDE can not be transformed to the 1st Lie canonical form which is identical to the classical heat equation.

Transformation: BDT-PDE to 2nd Lie canonical form

The BDT-PDE can be reduce to 2nd Lie canonical form if and only if the necessary and sufficient conditions in *Theorem 3* hold. These conditions reduce to the following case:

$$20L_x + 10\left[\sqrt{|a|}L_x\right]_x \int \frac{\mathrm{d}x}{\sqrt{|a|}} + \left[\sqrt{|a|}\left[\sqrt{|a|}L_x\right]_x\right]_x \left(\int \frac{\mathrm{d}x}{\sqrt{|a|}}\right)^2 = 0 \tag{17}$$

since M and N are 0. If we evaluate each term of this equation and substitute them into of it, we will obtain the equivalent equation:

$$20\left(\frac{1}{2}\sigma^2\right) + 10\left(\frac{\sigma^3}{2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{\sigma}\ln x + c_1\right) + \frac{\sigma^4}{4}\left(\frac{\sqrt{2}}{\sigma}\ln x + c_1\right)^2 = 0 \tag{18}$$

After some simplifications and choosing $c_1 = 0$ for the simplicity then we have:

$$20\sigma^2 + 10\sigma^2 \ln x + \sigma^2 \left(\ln x\right)^2 = 0, \quad \sigma \neq 0$$

It is nothing more than:

$$\left(\ln x\right)^2 + 10\ln x + 20 = 0 \tag{19}$$

This equation holds only for two x values (*i. e.* $x_1 = \mathrm{e}^{2\sqrt{5}-5}$ and $x_2 = \mathrm{e}^{-2\sqrt{5}-5}$). For this reason, BDT-PDE can not be reduced almost to the 2nd Lie canonical form.

Transformation: BDT-PDE to 3rd Lie canonical form

The linear parabolic (1+1) PDE (2) can be transformed to the 3rd Lie canonical form since the BDT-PDE (2) does not satisfy condition (8) in *Theorems 2* and condition (11) in *Theorem 3*. Then, the transformations from BDT-PDE to the 3rd Lie canonical form can be found if and only if the statements of the *Theorem 4* hold. Now, we are in the position to investigate the validation of *Theorem 4* for eq. (2). If we evaluate the necessary terms in eq. (12) then the following equation appears:

$$\frac{\ln x}{\sigma^2} d''(t) + \left[\frac{d(t)}{\sigma^2} - \frac{1}{2}\right] d'(t) = 0 \tag{20}$$

It is clear that eq. (20) holds if and only if $d(t) = \lambda$ where the λ is a constant. In case, we can find the transformations from linear parabolic (1+1) PDE (2) to 3rd Lie canonical form

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \bar{c}\left(\bar{x}\right)\bar{u}$$

by using eq. (13) in *Theorem 4*. After some calculations we obtain the following transformations in the barred co-ordinates:

$$\overline{t} = -t + a_1, \quad \epsilon = -1, \quad a_1 = \text{const.}$$

$$\overline{x} = \pm\frac{\sqrt{2}}{\sigma}\ln x + c_2 \tag{21}$$

$$\overline{u} = \upsilon_0\left|-\frac{1}{2}\sigma^2 x^2\right|^{-0.25} u\,\exp\left(\frac{\lambda}{\sigma^2}\ln x + c_3\right)$$

where

$$\overline{c}\left(\overline{x}\right) = \overline{x} - \frac{1}{2}\lambda + \frac{1}{8}\sigma^2 + \frac{1}{2}\frac{\lambda^2}{\sigma^2} \tag{22}$$

We are very close to find the analytic solution of BDT-PDE with this transformations which represent the relationship of the solutions between the barred co-ordinate $\overline{u}$ to the Cartesian co-ordinate u. Therefore, we need to move one step further to reach the result. Then, we solve the $\overline{u}_{\overline{t}} = \overline{u}_{\overline{x}\overline{x}} + \overline{c}(\overline{x})\overline{u}$ via methods of separation of variables (see *Appendix* for the details of the calculations) and obtain the following solution $\overline{u}$ in the barred co-ordinate:

$$\overline{u} = \left[c_0 X_1(\overline{x}) + c_1 X_2(\overline{x})\right]\mathrm{e}^{-k\overline{t}} \tag{23}$$

If we substitute eq. (23) in the eq. (21) then we obtain analytic solution of BDT-PDE:

$$u(x,t) = \upsilon_0^{-1}\left|-\frac{1}{2}\sigma^2 x^2\right|^{0.25}\exp\left[-\left(\frac{\lambda}{\sigma^2}\ln x + c_3\right)\right]\left[c_0 X_1(\overline{x}) + c_1 X_2(\overline{x})\right]\mathrm{e}^{-k\overline{t}} \tag{24}$$

where

$$X_1(\overline{x}) = 1 - \frac{\mu}{2!}\overline{x}^2 - \frac{1}{3!}\overline{x}^3 + \frac{\mu^2}{4!}\overline{x}^4 + \frac{1+3\mu}{5!}\overline{x}^5 + \frac{\mu^3 - 4}{6!}\overline{x}^6 + \cdots$$

$$X_2(\overline{x}) = \overline{x} - \frac{\mu}{3!}\overline{x}^3 - \frac{2}{4!}\overline{x}^4 + \frac{\mu^2}{5!}\overline{x}^5 + \frac{6\mu}{6!}\overline{x}^6 + \cdots$$

and

$$\overline{t} = -t + a_1 \text{ and } \overline{x} = \pm\frac{\sqrt{2}}{\sigma}\ln x + c_2$$

Transformation: BDT-PDE to 4th Lie canonical form

If $d(t) \neq \lambda$ (constant) in eq. (2) then it does not satisfy the conditions of *Theorem 2-4* so that it is reducible to the fourth Lie canonical form:

$$\frac{\partial\overline{u}}{\partial\overline{t}} = \frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \overline{c}\left(\overline{x},\overline{t}\right)\overline{u}$$

with respect to *Theorem 5*. We investigate solution of the following equation:

$$\frac{\ln x}{\sigma^2}d''(t) + \left[\frac{d(t)}{\sigma^2} - \frac{1}{2}\right]d'(t) \neq 0 \tag{25}$$

For simplicity of calculation, we can choose $d(t) = t$ without loss of generality then eq. (25) reduce to:

$$\frac{t}{\sigma^2}-\frac{1}{2}\neq 0 \tag{26}$$

This is also consistent with our assumption for $d(t)$. The $\overline{c}\left(\overline{x},\overline{t}\right)\overline{u}$ in the 4th Lie canonical form can be obtained:

$$\overline{c}\left(\overline{x},\overline{t}\right)=\overline{x}-\frac{1}{2}\overline{t}+\frac{\overline{t}^2}{2\sigma^2}+\frac{1}{8}\sigma^2+\frac{\ln\overline{x}}{\sigma^2}$$

by using the formula which is defined in *Theorem 4*. We then have:

$$\frac{\partial\overline{u}}{\partial\overline{t}}=\frac{\partial^2\overline{u}}{\partial\overline{x}^2}+\left(\overline{x}-\frac{1}{2}\overline{t}+\frac{\overline{t}^2}{2\sigma^2}+\frac{1}{8}\sigma^2+\frac{\ln\overline{x}}{\sigma^2}\right)\overline{u}$$

If we apply methods of separation of variables then we obtain:

$$\frac{T'}{T}+\frac{1}{2}\overline{t}-\frac{\overline{t}^2}{2\sigma^2}+\frac{1}{8}\sigma^2=-k \tag{27}$$

$$X''+\left(x+\frac{\ln x}{\sigma^2}+k\right)=0 \tag{28}$$

where the k is a separation of variable constant. The eq. (27) can be easily integrated but the eq. (28) can not be integrated as much as easy of other. It is really hard to obtain the analytical solution of eq. (28). On the other hand, its numerical solution may be possible to obtain by using some package programs (*i. e.* MATHEMATICA, *etc.*) As a result of the fact that, our aim in this paper is finding the analytical solution of BDT-PDE for the corresponding Lie canonical form, for this purpose we present that it can be reduced to the 4th Lie canonical form but it is not possible to obtain analytical solution of it in this form, easily.

Conclusions

In the stochastic world, it is important to talk about analytic solution of the SDE being considered which is generally hard to obtain. For this purpose, we first convert BDT stochastic interest rate model into the parabolic PDE using the stochastic calculus tools. Then, we present some conditions and show how to convert it into the different Lie canonical forms under the invariant approaches.

We exhibit that it is not possible to obtain the transformations between BDT-PDE and first Lie canonical form which is classical heat equation. Therefore, we investigate the possibility of obtaining the transformations for the second Lie canonical form but we present that it can not be reduced to the second Lie canonical form in general except to the some special x values. Moreover, we achieve to get the transformations for reduction to the third Lie canonical form and we obtain analytical solution of BDT-PDE under the light of the theorems. After that, under the restriction of the drift term $d(t)\neq 0$, we prove that BDT-PDE can also be reduced to the fourth Lie canonical form theoretically. We conclude that its analytical solution is not possible to obtain easily. For this complexity, it would be desirable to obtain analytical solution of the BDT-PDE into the fourth Lie canonical form but we have not been able to do this and it is going to be a research subject of the another paper study.

Appendix

The 3rd Lie canonical form:

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \bar{c}\left(\bar{x}\right)\bar{u} \tag{29}$$

can be solved analytically via methods of separation of variables as following.

Let the solution $\bar{U}$ be defined as *XT* then $\bar{U}_{\bar{t}} = XT'$, $\bar{U}_{\bar{x}} = X'T$, and $\bar{U}_{\bar{x}\bar{x}} = X''T$. If we substitute these in eq. (29) then we get:

$$XT' = X''T + c\left(\bar{x}\right)XT$$

$$\frac{T'}{T} = \frac{X'' + c\left(\bar{x}\right)X}{X}$$

$$\frac{T'}{T} = \frac{X''}{X} + c\left(\bar{x}\right) = -k, \quad k = \text{constant}$$

Now, we have two equations which are need to be solved:

$$T' + kT = 0 \tag{30}$$

$$X'' + \left[c\left(\bar{x}\right) + k\right] = 0 \tag{31}$$

The solution of the eq. in (30) is $T = \mathrm{e}^{-k\bar{t}}$ which can be obtained easily. After that if we substitute

$$\bar{c}\left(\bar{x}\right) = \bar{x} - \frac{1}{2}\lambda + \frac{1}{8}\sigma^2 + \frac{1}{2}\frac{\lambda^2}{\sigma^2}$$

which is obtained in section *Analytic solution of Black-Derman Toy linear (1 + 1) parabolic PDE* by using *Theorem 4*, into eq. (31) then we have:

$$X'' + \left(\bar{x} - \frac{1}{2}\lambda + \frac{1}{8}\sigma^2 + \frac{1}{2}\frac{\lambda^2}{\sigma^2} + k\right)X = 0$$

Let as define:

$$\mu = -\frac{1}{2}\lambda + \frac{1}{8}\sigma^2 + \frac{1}{2}\frac{\lambda^2}{\sigma^2}$$

and rewrite previous equation in terms of it:

$$X'' + \left(\bar{x} + \mu\right)X = 0 \tag{32}$$

We can obtain solution of eq. (32) via power series approximation [12], for this purpose we define $X = \sum_{n=0}^{\infty} c_n \bar{x}^n$. We also generate $X' = \sum_{n=1}^{\infty} n c_n \bar{x}^{n-1}$, and $X' = \sum_{n=2}^{\infty} n\left(n-1\right) c_n \bar{x}^{n-2}$ by taking derivatives of *X* with respect to $\bar{x}$. Now, if we substitute these in eq. (32) it becomes:

$$\sum_{n=0}^{\infty}\left(n+2\right)\left(n+1\right)c_{n+2}\bar{x}^n + \sum_{n=1}^{\infty}c_{n-1}\bar{x}^n + \mu\sum_{n=0}^{\infty}c_n\bar{x}^n = 0$$

After some simplifications on this equation, we have:

$$2c_2 + \mu c_0 + \sum_{n=0}^{\infty}\left[\left(n+2\right)\left(n+1\right)c_{n+2} + c_{n-1} + \mu c_n\right]\bar{x}^n = 0 \tag{33}$$

This equation holds if and only if $2c_2 + \mu c_0 = 0$ and $(n+2)(n+1)c_{n+2} + c_{n-1} + \mu c_n = 0$ where $n = 1,2,3,...$ The latter equation can be written:

$$c_{n+2} = -\frac{c_{n-1} + \mu c_n}{(n+1)(n+2)}, \quad n = 1,2,3,\ldots$$

Now, we are able to determine all coefficients c_n is in terms of c_0 and c_1 for $n \geq 3$. First, we assume that $c_0 \neq 0$ and $c_1 = 0$ for simplicity. In this case, solutions are defined in terms of c_0. In second case, if we assume that $c_0 = 0$ and $c_1 \neq 0$ then the solutions can be defined as a function of c_1.

The coefficients in the first case when $c_0 \neq 0$ and $c_1 = 0$ are:

$$c_2 = -\frac{\mu}{2!}c_0\,,\;\; c_3 = -\frac{\mu}{3!}c_0\,,\;\; c_4 = \frac{\mu^2}{4!}c_0\,,\;\; c_5 = \frac{1+3\mu}{5!}c_0\,,\;\; c_6 = \frac{\mu^3 - 4}{6!}c_0\,,\;\cdots$$

The coefficients in the second case when $c_0 = 0$ and $c_1 \neq 0$ are:

$$c_2 = -\frac{\mu}{2!}0 = 0\,,\;\; c_3 = -\frac{\mu}{3!}c_1\,,\;\; c_4 = -\frac{2}{4!}c_1\,,\;\; c_5 = \frac{\mu^2}{5!}c_1\,,\;\; c_6 = \frac{6\mu}{6!}c_1\,,\;\cdots$$

The general solution is:

$$X\left(\bar{x}\right) = c_0 X_1\left(\bar{x}\right) c_1 X_2\left(\bar{x}\right) \tag{34}$$

where

$$X_1\left(\bar{x}\right) = 1 - \frac{\mu}{2!}\bar{x}^2 - \frac{1}{3!}\bar{x}^3 + \frac{\mu^2}{4!}\bar{x}^4 + \frac{1+3\mu}{5!}\bar{x}^5 + \frac{\mu^3-4}{6!}\bar{x}^6 + \cdots$$

$$X_2\left(\bar{x}\right) = \bar{x} - \frac{\mu}{3!}\bar{x}^3 - \frac{2}{4!}\bar{x}^4 + \frac{\mu^2}{5!}\bar{x}^5 + \frac{6\mu}{6!}\bar{x}^6 + \cdots$$

The solution of the 3rd Lie canonical form was defined as $\bar{U}(\bar{x},\bar{t}) = X(\bar{x})T(\bar{t})$, if we replace $X(\bar{x})$ and $T(\bar{t})$ with $c_0 X_1(\bar{x}) c_1 X_2(\bar{x})$ and $\mathrm{e}^{-k\bar{t}}$, respectively, then we have the solution:

$$\bar{U}\left(\bar{x},\bar{t}\right) = \left[c_0 X_1\left(\bar{x}\right) c_1 X_2\left(\bar{x}\right)\right]\mathrm{e}^{-k\bar{t}} \tag{35}$$

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