A HEURISTIC OPTIMIZATION METHOD OF FRACTIONAL CONVECTION REACTION
An Application to Diffusion Process

by

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The convection differential models play an essential role in studying different chemical process and effects of the diffusion process. This paper intends to provide optimized numerical results of such equations based on the conformable fractional derivative. Subsequently, a well-known heuristic optimization technique, differential evolution algorithm, is worked out together with the Taylor's series expansion, to attain the optimized results. In the scheme of the Taylor optimization method (TOM), after expanding the functions with the Taylor's series, the unknown terms of the series are then globally optimized using differential evolution. Moreover, to illustrate the applicability of TOM, some examples of linear and non-linear fractional convection diffusion equations are exemplified graphically. The obtained assessments and comparative demonstrations divulged the rapid convergence of the estimated solutions towards the exact solutions. Comprising with an effective expander and efficient optimizer, TOM reveals to be an appropriate approach to solve different fractional differential equations modeling various problems of engineering.

Key words: *conformable fractional derivative, Taylor's series, optimization*

Introduction

Modeling dissimilar real-world phenomena using fractional definitions have become the most highly desiring areas of realistic sciences. For the reason that the non-local properties of fractional operator enable these differential models to put the information, about the recent and the historical situation, in a nutshell [1]. For the last decades, many enlargements have been made in this regard to explore enhanced definitions and properties in order to overcome the inadequacies of previous definitions of fractional calculus, such as, He's fractional derivative [3], Atangana-Baleanu fractional derivative [4], Caputo and Fabrizio [5], conformable derivative [6], *etc.* Consequently, these novel aspects enrich the capabilities of fractional differential models in bringing diverse physical significances to light [7-9]. Hence, by means of different theories of fractional derivatives, the behaviors of many fractional PDE have been studied and various techniques have been developed [10-14] but still there are many thinks to be done in this direction.

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The fractional diffusion equations are mostly used in relating abnormal slowly-diffusion phenomenon and describing the abnormal convection phenomenon of liquid in the medium. It is broadly applied in engineering and science, as mathematical models that are used to replicate computing. Numerous numerical methods have been designed in this regard. In [15], differential transform method has been developed for fractional convection-diffusion equation. A new variational iteration method was applied to decipher fractional convection-diffusion equations in [16]. Wavelets based method has been considered in [17] to solve the space-time fractional convection-diffusion equations. An operational matrix of fractional-order Legendre functions has been employed to solve the time fractional convection diffusion equation in [18]. Furthermore, a lot of work has been discussed for the solution of fractional diffusion equations [19-22].

In this manuscript, employing useful properties of Taylor's series with the heuristic optimization techniques, we consider the following types of time fractional convection-diffusion equation:

$$\frac{\partial^\lambda z}{\partial \tau^\lambda} = \frac{\partial^2 z}{\partial x^2} - A\frac{\partial z}{\partial x} + N(z) + G(x,\tau), \quad a \le x \le b, \quad \tau \ge 0, \quad 0 < \lambda \le 1 \tag{1}$$

subject to the initial-boundary conditions:

$$z(x,0) = q_0(x), \quad z(a,\tau) = s_0(\tau), \quad z(b,\tau) = s_1(\tau) \tag{2}$$

where $N(z)$ is a non-linear operator representing the potential energy, A is a fixed parameter, and λ denotes the fractional order derivative. The physical understanding of eq. (1) in thermal engineering represents temperature or species concentration of the heat or mass transferred inside the system due to diffusion and convection term.

In literature, the heuristic optimization techniques have leaded the way to obtain solutions of fractional differential equations. These global optimizing programs play a crucial part in assessing quick and accurate approximate solutions of the differential models, in graphical as well as in tabular representation. Here, we have utilized differential evolution (DE) algorithm [22-24], which is related to the natural processes and genetics of populations. For the most important part of DE algorithm is that it requires objective functions and its derivatives to be continuous and makes a high number of evaluations of the objective function. Thus, for the governing problem, an error function is constructed by means of Taylor's series expansion of the related functions as the objective function.

The Taylor's series expansion, having the effective ability to approximate the functions more accurately than other polynomial approximations. It has been widely used to acquire the solutions of an integer as well as fractional differential equations [25-26]. The amalgamation of Taylor's series expansion and DE algorithm, name as TOM, considered in this attempt, exhibits a remarkable tool to acquire the effective solutions of functions together with the globally optimized values of the error function. In addition to this, the illustrative examples, considered with conformable fractional derivative [5, 6, 10], elevated the efficiency, stability and appropriateness of TOM.

Fractional background

Conformable fractional derivative

For any function $z:[0,\infty) \to \Re$, the conformable fractional derivative of order α is given [5]:

$$\mathcal{T}_\alpha z(\tau) = \lim_{\hbar \to 0} \frac{z\left(\tau + \hbar \tau^{1-\alpha}\right) - z(\tau)}{\hbar} \tag{3}$$

for all $\tau > 0$, $\alpha \in (0,1]$. In addition, if, is ($\lceil \alpha \rceil + 1$)-differentiable and continuous at, then four $\alpha \in (\lceil \alpha \rceil, \lceil \alpha \rceil + 1]$:

$$\mathcal{T}_\alpha z(\tau) = \lim_{\hbar \to 0} \frac{z^{(\lceil \alpha \rceil - 1)}\left(\tau + \hbar \tau^{(\lceil \alpha \rceil - \alpha)}\right) - z^{(\lceil \alpha \rceil - 1)}(\tau)}{\hbar} \tag{4}$$

For further details and proofs, one would see [5, 6].

Taylor optimization method

Consider a continuous and differentiable function $z(x,\tau): \Re \to \Re$, define over an interval $[0,1) \times [0,1)$, then for any integer $\vartheta > 0$ the subsequent equality holds:

$$z(x,\tau) = w_0 + v_1 x + w_1 \tau + AF(x,\tau) \tag{5}$$

where $v_k = z_x^{(k)}(x,\tau)$ and $w_k = z_\tau^{(k)}(x,\tau)$ are the k^{th} order derivative (for $k = 1,2,...,N$) of $z(x,\tau)$ at $x = 0$ and $\tau = 0$, respectively, and

$$AF(x,\tau) = \sum_{k=2}^{N} \frac{\left(v_k x + w_k \tau\right)^k}{k!}$$

which represents the truncated Taylor's series.

Here, we concern with the non-linear fractional initial-boundary value problem, eqs. (1) and (2), with conformable fractional definition with fractional order $0 < \lambda \le 1$:

$$\mathcal{T}_\lambda z = z_{xx} - z_x + N(z) + G(x,\tau) \tag{6}$$

where $\mathcal{T}_\lambda$ represents the conformable fractional operator. This model has significant importance in many physical situations [15-17]. Hence, the graphical interpretations and optimized solutions of the non-linear fractional model of eq. (6) is the focal point of this attempt. Hence, in order to assess the solutions, we employ the Taylor's optimization method. This method initiates with the construction of trial solutions of the unknown functions, on using truncated Taylor's series expansion as defined in eq. (5) with the given initial conditions. Thus, the trial solutions $\hat{z}_{\text{trial}}(x,\tau)$ of eq. (6) can be expressed:

$$\hat{z}_{\text{trial}}(x,\tau,\Phi) = w_0 + v_1 x + w_1 \tau + AF(x,\tau) \tag{7}$$

where w_0, v_1, and w_1 are the initial and boundary values and Φ is a vector of v_k and w_k which are defined as the k^{th} derivative of N^{th} function at $x = 0$ and $\tau = 0$, that are to be determined. Now, to compute the remaining terms of Taylor's series expansion for each of these trial solutions, we set up an error function, which is then optimized by using an optimizing technique.

In this endeavor, for the optimization purpose the DE algorithm is utilized. This effective heuristic optimizing technique was proposed in [23]. Among many other global optimizers, DE is considered to be more significant for its simplicity and strong population-based stochastic search technique over a continuous domain. The key features of DE are the three control parameters, *i. e.*, the population size *NP*, cross-over constant *CR* and the scaling factor *Sf*. These parameters may extensively affect the optimization performance of the DE, therefore, in

[22-24] some simple rules are defined for the selection of these parameters. The DE algorithm maintains the population of NP-dimensional parameter vectors, known as individuals, which makes new applicant solutions by taking the parent individual and several other individuals of the same population. Using the mutation operation, it randomly picks the generated vectors from the population to produce a mutant vector, which is said to be the target vector. The initial population is established:

$$v_j^L + \text{rand}(0,1)\left(v_j^U - v_j^L\right) \tag{8}$$

After the completion of mutation operation, cross-over phase is considered in which each pair of target vector is taken with its corresponding mutant vector to generate a trail vector. After the cross-over operation, selection of the function value is performed by comparing the function value of each trial vector to that of its corresponding target vector in the current population. If the function value of each trial vector becomes less than or equal to the target vector, then the trial vector will change the target vector and comes in the population of the next generation. The selection operator process can be expressed:

$$v_i(t+1) = \begin{cases} u_i(t) & \text{if} \quad F\left[u_i(t)\right] \le F\left[v_i(t)\right] \\ v_i(t) & \text{if} \quad F\left[u_i(t)\right] > F\left[v_i(t)\right] \end{cases} \tag{9}$$

where $u_i(t)$ and $v_i(t)$ are the trial and target vectors, respectively, and F is the objective function. The process is considered to be convergent if the best function values, in the new and old populations, with the new best point and the old best point, have less difference than the tolerance level.

Thus, in DE algorithm, the solutions are easily obtained by just specifying the population set, trial solution and the objective function. For the governing problems, the objective function is defined by using the error functions, E, for the conformable fractional operator:

$$E = -\mathbf{T}\dot{\mathbf{Z}} + \mathbf{Z}'' - \mathbf{Z}' + \mathbf{N} + \mathbf{G} \tag{10}$$

where dot and prime represent the partial derivatives of the functions with respect to τ and x, respectively, which can be expressed in matrix form:

$$\boldsymbol{Z} = \left[\hat{z}(x_1,\tau_1;\Phi) \quad \hat{z}(x_2,\tau_2;\Phi) \quad \ldots \quad \hat{z}(x_m,\tau_m;\Phi)\right]^t$$

$$\boldsymbol{Z}' = \left[\hat{z}'(x_1,\tau_1;\Phi) \quad \hat{z}'(x_2,\tau_2;\Phi) \quad \ldots \quad \hat{z}'(x_m,\tau_m;\Phi)\right]^{t'}$$

$$\dot{\boldsymbol{Z}} = \left[\dot{\hat{z}}(x_1,\tau_1;\Phi) \quad \dot{\hat{z}}(x_2,\tau_2;\Phi) \quad \ldots \quad \dot{\hat{z}}(x_m,\tau_m;\Phi)\right]^t$$

$$\boldsymbol{N} = \left\{N\left[\hat{z}(x_1,\tau_1;\Phi)\right] \quad N\left[\hat{z}(x_2,\tau_2;\Phi)\right] \quad \ldots \quad N\left[\hat{z}(x_m,\tau_m;\Phi)\right]\right\}^t$$

$$\boldsymbol{G} = \left[G(x_1,\tau_1;\Phi) \quad G(x_2,\tau_2;\Phi) \quad \ldots \quad G(x_m,\tau_m;\Phi)\right]^t$$

$$\boldsymbol{T} = \begin{bmatrix} \tau_1^{1-\lambda} & 0 & \cdots & 0 \\ 0 & \tau_2^{1-\lambda} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \tau_m^{1-\lambda} \end{bmatrix}$$

And the selection operator process is expressed:

$$\frac{1}{m^2}\min\left(E^2\right)\le 10^{-\eta}$$

for the population set $(0,m]$, where η is any positive integer.

The algorithm

Accordingly, the algorithmic process of TOM for immediate implementation on convection advection eq. (1) is outlined:

$$z(x,\tau)=z(x,\tau)+v_1x+w_1\tau+AF(x,\tau)$$

Step 1. (i) Set $N\ge 2$. (ii) Find w_0, w_1, v_1 by using $z_0=q_0(x)$, $z_a=s_0(\tau)$, $z_b=s_1(\tau)$, and (iii) Construct the trial solution:

$$\hat{z}_{\text{trial}}(x,\tau,\Phi)=z_0-AF(0,\tau)-AF(x,0)+$$
$$+\frac{x}{b}\left[z_b-z_a-AF(0,\tau)-AF(b,\tau)+AF(x,0)\right]+AF(x,\tau;\Phi)$$

Step 2. (i) Set $0<\lambda\le 1$ and (ii) compute all the components of given advection, *i. e.*

$$\mathcal{T}_\lambda Z_{\text{trial}}(x,\tau,\Phi)=\tau^{1-\lambda}D_\tau\hat{z}_{\text{trial}}(x,\tau,\Phi),\quad Z_{xx}=D_{xx}\hat{z}_{\text{trial}}(x,\tau,\Phi)\quad Z_x=D_x\hat{z}_{\text{trial}}(x,\tau,\Phi)$$

Step 3. (i) Set sampling points:

$$x_i=\tau_i=\frac{(b-a)i}{m},\qquad i=0,1,\ldots,m \text{ for } m\ge N \text{ and}$$

(ii) Substitute computed component and the trial solution in error function, defined in eq. (10):

$$\min\ E(\Phi)=\frac{1}{m^2}\sum_{i=1}^{m}\left\{\begin{matrix}-\tau_i^{1-\lambda}D_\tau\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi)+D_{xx}\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi)-\\-D_x\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi)+N\left[\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi)\right]+G(x_i,\tau_i)\end{matrix}\right\}^2$$

Step 4. Input: Fit the error functions in the DE algorithm. On some manipulation, effective values of each terms of Φ along with a global minimum value of mean square of error functions are attained by using MATHEMATICA software.

Output: Global minimum value of E and the values of all unknown terms in Φ.

Step 5. Input: Substitute the values of Φ in trial solutions $\hat{z}_{\text{trial}}(x,\tau)$.

Output: Approximate solutions of $\hat{z}_{\text{trial}}(x,\tau)$.

Testing the algorithm

Test problem 1. Consider the initial-boundary problem of time-fractional convection-diffusion equation [15, 16] defined in eq. (1) with:

$$N(z)=0,\quad A=x,\quad G(x,\tau)=2\tau^\lambda+2x^2+2,\quad 0<x<1,\quad 0<\tau<1 \tag{11}$$

and initial-boundary conditions:

$$z(x,0)=x^2,\quad z(0,\tau)=2\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}t^{2\lambda},\quad z(1,\tau)=1+2\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}t^{2\lambda} \tag{12}$$

Here, $\Gamma(\cdot)$ is the gamma function. The exact solution of eq. (11) is given:

$$z(x,\tau) = x^2 + 2\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}t^{2\lambda}$$

Taking $N = 6$ trial solution of eqs. (11) and (12), can be computed by setting:

$$z_0 = x^2, \quad z_a = 2\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}t^{2\lambda}, \quad z_b = 1 + 2\frac{\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}t^{2\lambda} \tag{13}$$

The constructed trial solution of eqs. (12) and (13) can be written:

$$\begin{aligned} \hat{z}_{\text{trial}}(x,\tau,\Phi) &= x^2 - AF(0,\tau;\Phi) - AF(x,0;\Phi) + AF(x,\tau;\Phi) + \\ &+ \frac{x}{b}\left[1 - AF(0,\tau;\Phi) - AF(b,\tau;\Phi) + AF(x,0;\Phi)\right] \end{aligned} \tag{14}$$

Equation (14) helps us to compute all components of eq. (6). Using definition given in section *Conformable fractional derivative*, and substitute all of these in the residual error function:

$$\min\ E(x_i,\tau_i,\Phi) = \frac{1}{m^2}\sum_{i=1}^{m}\begin{Bmatrix} -\tau_i^{1-\lambda} D_\tau \hat{z}_{\text{trial}}(x_i,\tau_i;\Phi) + D_{xx}\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi) - \\ -D_x\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi) + N\left[\hat{z}_{\text{trial}}(x_i,\tau_i;\Phi)\right] + G(x_i,\tau_i) \end{Bmatrix}^2 \tag{15}$$

where $\tau_i, x_i \in [0,1]$. Now, on implementing DE algorithm the mean square error in eq. (15) is globally minimized and approximate solution is obtained. Here, we consider six terms of Taylor's series expansion, *i. e.* $N = 6$, and population size $NP = 20$, from the population set [0,1], to acquire the graphical and tabulated solutions of $\hat{z}_{\text{trial}}(x,\tau)$ at various values of λ. Sequentially, fig. 1(a) displays the comparison of the exact solution with an approximate solution of $\hat{z}_{\text{trial}}(x,\tau)$ with conformable fractional operator $\lambda = 1$. The absolute error of the function $\hat{z}_{\text{trial}}(x,\tau)$ is plotted in fig. 3(a). Additionally, comparative explanation of the proposed algorithm with the previous method [15] is exhibited in tab. 1 for different values of λ. Moreover, some numerical approximations are also plotted in fig. 2(a) for different values of λ, in order to demonstrate the effects of fractional operator on the solution.

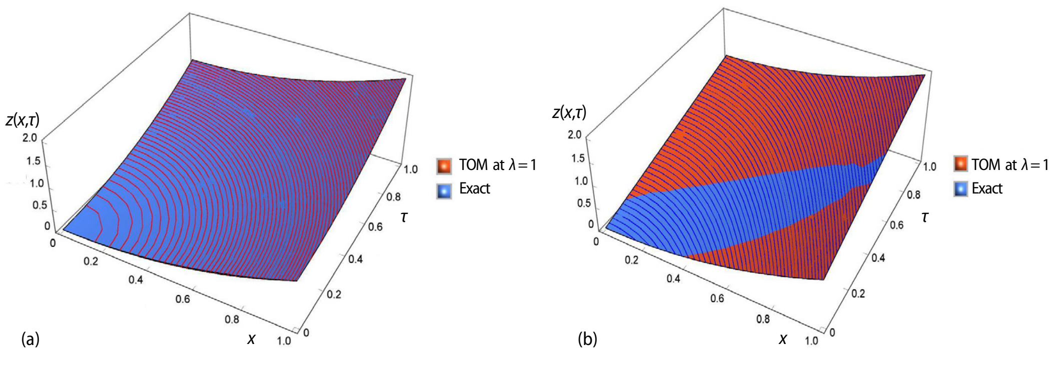

Figure 1. Solution of $z(x,\tau)$ (a) test problem 1 and (b) test problem 2 (*for color image see journal web site*)

Test problem 2. We consider the homogeneous fractional convection-diffusion equation [16,18] with the following values of the functions and the parameters:

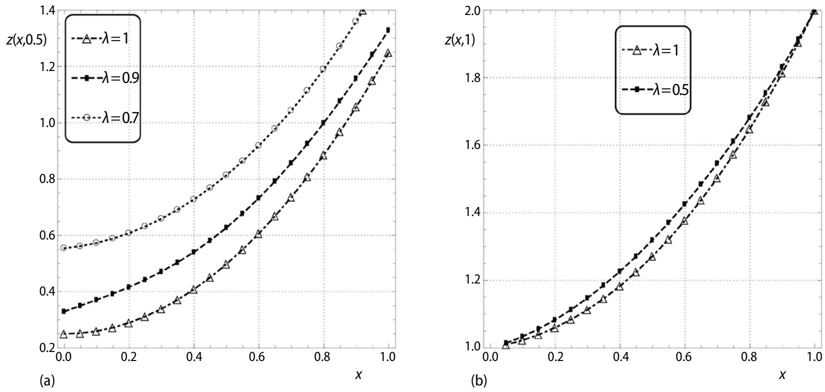

Figure 2. Diffusion behaviors at $\tau = 0.5$ (a) test problem 1 and (b) test problem 2

Table 1. Comparison of $\hat{z}(x,1)$ and the method in [11] for test problem 1

x	$\lambda = 1$		$\lambda = 0.9$		$\lambda = 0.7$	
	TOM	[11]	TOM	[11]	TOM	[11]
0.25	1.05279	1.0625	1.39648	1.20986	1.61654	1.52549
0.35	1.11316	1.1225	1.46966	1.26986	1.69338	1.58549
0.55	1.29666	1.3025	1.61686	1.44986	1.87376	1.76549
0.75	1.56032	1.5625	1.80556	1.70986	2.0973	2.02549
0.85	1.72151	1.7225	1.92625	1.86986	2.23005	2.18549
1	2	2	2.14736	2.14736	2.46299	2.46299

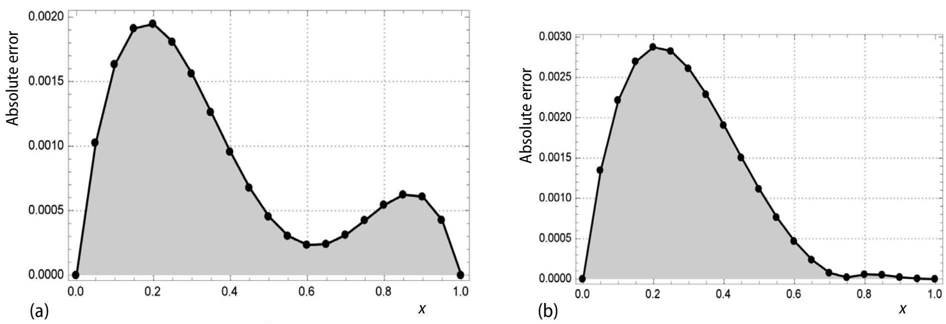

Figure 3. Absolute error at $\lambda = 1$ and $\tau = 0.5$ (a) test problem 1 and (b) test problem 2

$$N(z) = z_x z - 2xz, \quad A = 1, \quad G(x,\tau) = 2x - 2 + \Gamma(\lambda + 1), \quad 0 < x < 1, \quad 0 < \tau < 1 \tag{16}$$

with initial and the boundary conditions:

$$z(x,0) = x^2, \quad z(0,\tau) = \tau^\lambda, \quad z(1,\tau) = 1 + \tau^\lambda \tag{17}$$

The exact solution of the problem is $z(x,\tau) = x^2 + \tau^\lambda$. Taking $N = 6$ the trial solution of eqs. (16) and (17), can be computed by setting $z_0(x) = x^2$, $z_a(\tau) = t^\lambda$, $z_b(\tau) = 1 + t^\lambda$,and we have the same trial solution in eq. (14).

After substituting all components of eq. (6) the residual error function:

$$\min\ E\left(x_i,\tau_i,\Phi\right)=\frac{1}{m^2}\sum_{i=1}^{m}\left\{\begin{array}{l}-\tau_i^{1-\lambda}D_\tau\hat{z}_{\text{trial}}\left(x_i,\tau_i;\Phi\right)+D_{xx}\hat{z}_{\text{trial}}\left(x_i,\tau_i;\Phi\right)-\\-D_x\hat{z}_{\text{trial}}\left(x_i,\tau_i;\Phi\right)+N\left[\hat{z}_{\text{trial}}\left(x_i,\tau_i;\Phi\right)\right]+G\left(x_i,\tau_i\right)\end{array}\right\}^2 \tag{18}$$

The time-fractional convection-diffusion equation mentioned in eq. (14) have also been studied in [18], where fractional order Legendre function method is utilized to attain the approximate solution. Here, we use TOM to attain the optimum solutions at different values of λ. Figure 3(a) reveal the significant precision of approximate solution with the exact solution. Following the schematic algorithm, the measured numerical solutions of $\hat{z}_{\text{trial}}(x,\tau)$, at different values of λ are shown in fig. 3(b). Additionally, comparative explanation of the proposed algorithm with the previous methods [16, 18] is exhibited in tab. 2.

Table 2. Comparisons of $\hat{z}(x,0.5)$ and the methods in [10, 12] for test problem 2

x	$\lambda = 1$			$\lambda = 0.7$		$\lambda = 0.5$		
	TOM	[12]	[10]	TOM	[10]	TOM	[12]	[10]
0.25	0.56532	0.5625	0.5625	0.69637	0.678072	0.79899	0.76960	0.769607
0.35	0.62478	0.6225	0.6225	0.76118	0.738072	0.86463	0.82960	0.829607
0.55	0.80326	0.8025	0.8025	0.94475	0.918072	1.04435	1.00961	1.00961
0.75	1.06248	1.0625	1.0625	1.19911	1.17807	1.2913	1.26961	1.26961
0.85	1.22245	1.2225	1.2225	1.35259	1.33807	1.44222	1.42961	1.42961
1	1.5	1.5	1.5	1.61557	1.61557	1.70711	1.70711	1.70711

Conclusions

In this work, time-fractional convection diffusion equations were analyzed with conformable fractional derivatives. We deliberated the optimized solutions by means of TOM. The manifestation given in section *Taylor optimization method* and the ascertained values of $E \leq 10^{-k}$ for positive integers $k = 1,2...27$ in tab. 3 signify the worth mentioning accuracy of the proposed approach. Thus, from the facts and figures, it is possible to conclude as follows.

Table 3. Glob al optimum error values for $NP = 20$ and $N = 6$

λ	Test problem 1	Test problem 2
1.0	$1.3408\cdot10^{-27}$	$3.5790\cdot10^{-11}$
0.9	$2.7722\cdot10^{-20}$	$7.3678\cdot10^{-10}$
0.7	$6.1530\cdot10^{-16}$	$1.2063\cdot10^{-10}$
0.5	$2.6231\cdot10^{-13}$	$1.5315\cdot10^{-9}$

- Modeling with conformable fractional derivative supported the physical meaning of the governing model and provided a new purse for modelling many problems of applied sciences.
- Just depending on the basic limit definition of the derivative, conformable fractional derivative is simple and can be easily used to execute fractional behaviors of the functions.
- To calculate the unknown terms of Taylor's series expansion, the optimizing algorithms give effective results by simply optimizing the error functions.
- As shown in tab. 3, the population based optimizing algorithm DE, gives the global optimum values of mean square errors of the test problems at different fractional values in a continuous domain, which verifies the appropriateness of approximated functions of the governing models.

Nomenclature

$AF(x,\tau)$ – truncated Taylor's series
$E(\Phi)$ – the residual error
m – real numbers
$N(z)$ – potential energy

v_k, w_k – weight vector
x_i, τ_i – sampling points
$\hat{z}_{\text{trial}}(x,\tau)$ – trial solution

Greek symbols

η – positive integer
λ – fractional order derivative
$\mathcal{T}_\alpha$ – conformable fractional operator
Φ – vector

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