FRACTAL DERIVATIVE MODEL FOR THE TRANSPORT OF THE SUSPENDED SEDIMENT IN UNSTEADY FLOWS

by

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This paper makes an attempt to develop a Hausdorff fractal derivative model for describing the vertical distribution of suspended sediment in unsteady flow. The index of Hausdorff fractal derivative depends on the spatial location and the temporal moment in sediment transport. We also derive the approximate solution of the Hausdorff fractal derivative advection-dispersion equation model for the suspended sediment concentration distribution, to simulate the dynamics procedure of suspended concentration. Numerical simulation results verify that the Hausdorff fractal derivative model provides a good agreement with the experimental data, which implies that the Hausdorff fractal derivative model can serve as a candidate to describe the vertical distribution of suspended sediment concentration in unsteady flow.

Key words: *Hausdorff fractal derivative, suspended sediment, scaling transformation, advection-dispersion*

Introduction

In the past few decades, the vertical distribution of suspended sediment concentration in unsteady flows has been an important topic in sediment transport mechanics [1-4]. Numerous literatures related to the vertical distribution of the suspended sediment concentration in unsteady flow, have been reported. Rouse [5] proposed a famous formula for vertical distribution of sediment concentration in steady flow, based on turbulent diffusion theory. Van Rijn [6] presented a 2-D vertical mathematical model for suspended sediment, which has been developed as a tool for routine morphological computations in the daily engineering practice. Then some scholars have further research on vertical distribution of suspended sediment concentration in unsteady flow [1, 3, 4, 7-11].

The suspended sediment transport equation is derived based on turbulent diffusion theory, which contains the most important roles played by advection, turbulent diffusion, and gravitation in the suspension of sediment particles [1]. The physical mechanism of an unsteady sediment suspension distribution is a dynamic of vertical fluxes between downward sediment

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settling and upward turbulent diffusion. Base on the theory of the turbulent diffusion, the vertical distribution of suspended sediment is an anomalous diffusion behavior [12-14].

In the previous study, different approaches have been proposed to anomalous diffusion, such as fractional derivative models [15-18], Hausdorff fractal derivative models [19-21], *etc.* Recently, fractal calculus and corresponding fractal PDE have paid increasing attention in various scientific fields involving heavy-tailed dynamics [20-23]. Then we employ the Hausdorff fractal derivative model as a local derivative model, which have been application in complex system, such as anomalous diffusion [22-25], viscoelastic materials [26], heat generation [27], and relaxation [28]. As an alternative method, the Hausdorff fractal derivative is based on time-space scale transforms. And the concept of the Hausdorff fractal derivative has been presented underlying Hausdorff dimensions [19]. Previous research illustrated that the Hausdorff fractal derivative model was simple compared with fractional derivative model and could suitably describe the complex behavior. However, the fractal structure usually changes with time or space and yields the anomalous diffusion behavior [29-32]. Therefore, in this study, we try to develop a new model to describe the vertical distribution of suspended sediment. We propose the time-space Hausdorff fractal derivative advection-dispersion model to depict the suspended sediment concentration. The index of Hausdorff fractal derivative model changes with time, space, corresponding to trapping effect of sediment transport and the vertical displacement of suspended sediment. The Hausdorff fractal derivative model provides a good agreement with the experimental data.

Hausdorff fractal derivative model

The definition of Hausdorff fractal derivative

When the standard integer derivative is replaced by Hausdorff fractal derivative, the definition of the Hausdorff fractal derivative can be stated [19]:

$$\begin{aligned}\frac{\partial S(y,t)}{\partial t^{\alpha}} &= \lim_{t_1 \to t} \frac{S(y,t_1) - S(y,t)}{t_1^{\alpha} - t^{\alpha}} \\ \frac{\partial S(y,t)}{\partial y^{\beta}} &= \lim_{y_1 \to y} \frac{S(y_1,t) - S(y,t)}{y_1^{\beta} - y^{\beta}}\end{aligned} \tag{1}$$

where α and β represent the order of the Hausdorff fractal derivative in time and space ($0 < \alpha \leq 1$, $0 < \beta \leq 1$), respectively.

Base on the hypothesis of the fractal invariance and equivalence, the Hausdorff fractal derivative can be restated as normal derivative by using the metric transforms [19]:

$$\begin{cases}\hat{t} = t^{\alpha} \\ \hat{y} = y^{\beta}\end{cases} \tag{2}$$

Hausdorff fractal derivative advection-dispersion equation

The Hausdorff fractal derivative advection-dispersion equation is an alternative approach to describe the anomalous diffusion, which can be expressed:

$$\frac{\partial S(y,t)}{\partial t^{\alpha}} = -\omega \frac{\partial S(y,t)}{\partial y^{\beta}} + \frac{\partial}{\partial y^{\beta}}\left[\varepsilon_{sy} \frac{\partial S(y,t)}{\partial y^{\beta}}\right] \tag{3}$$

where S is the suspended sediment concentration, the time Hausdorff fractal derivative term $\partial S/\partial t^\alpha$ captures the scale dependency of the sediment transport, and the space Hausdorff fractal derivative term $\partial S/\partial y^\beta$ describes the heavy tailed transport of sediment. The sediment transport along the flume can be simulated by solving the governing eq. (3). The ε_{sy} is a constant which denotes diffusion coefficient, and ω – the sediment settling velocity.

The following initial and boundary condition is imposed on:

$$S(y,0)=C, \quad \text{and} \quad \left(\omega S-\varepsilon_{sy}\frac{\partial S}{\partial y}\right)\Big|_{y=0}=0\,, \quad \frac{\partial S}{\partial y}\Big|_{y=h}=0$$

The approximate solutions are simple and accurate to evaluate and can be used to validate models for describing the physical problem based on advection-dispersion equation. According to the previous investigation on analytical solution on advection-dispersion equation, the approximate solution of eq. (3) is [33, 34]:

$$S(y,t)=C-$$

$$-C\left[\frac{1}{2}\operatorname{erfc}\left(\frac{y^\beta-\omega t^\alpha}{\sqrt{4\varepsilon_{sy}t^\alpha}}\right)+\sqrt{\frac{\omega^2 t^\alpha}{\pi\varepsilon_{sy}}}e^{-\frac{\left(y^\beta-\omega t^\alpha\right)^2}{4\varepsilon_{sy}t^\alpha}}-\frac{1}{2}\left(1+\frac{\omega y^\beta}{\varepsilon_{sy}}+\frac{\omega^2 t^\alpha}{\varepsilon_{sy}}\right)e^{-\frac{\omega y^\beta}{\varepsilon_{sy}}}\operatorname{erfc}\left(\frac{y^\beta+\omega t^\alpha}{\sqrt{4\varepsilon_{sy}t^\alpha}}\right)\right] \quad (4)$$

Results and discussion

The numerical examples of the Hausdorff fractal derivative advection-dispersion model *i. e.*, model (3), with a constant time index α and a constant space index β are shown in figs. 1 and 2. The time Hausdorff fractal derivative model can describe the suspended sediment transport from sub to normal diffusion. The space Hausdorff fractal derivative model can describe the suspended sediment transport from normal to super-diffusion. As shown in fig. 1(a), with the order of the time Hausdorff fractal derivative α decreasing, the diffusion rate

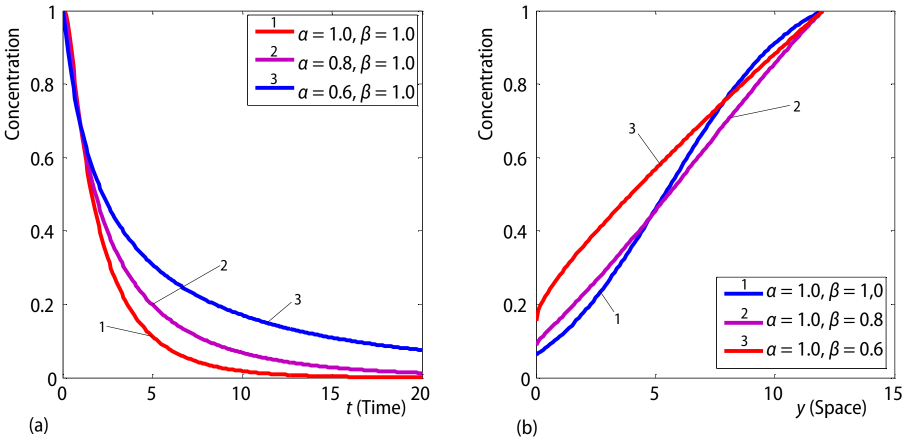

Figure 1. Dimensionless numerical results; (a) Dimensionless numerical simulation results of time Hausdorff fractal derivative advection-dispersion model, diffusion coefficient ε_{sy} = 1.0, the settling velocity ω = 1.0, and space Hausdorff fractal derivative order β = 1.0, y = 10, (b) Dimensionless numerical simulation results of space Hausdorff fractal derivative advection-dispersion model, diffusion coefficient ε_{sy} = 1.0, the settling velocity ω = 1.0, and time Hausdorff fractal derivative order α = 1.0, t = 12

of sediment change slower than normal diffusion, the curve of sediment contribution changes from steep to slow. It can be explained that the time correlation of Hausdorff fractal advection-dispersion equation can describe the memory effect of anomalous sediment transport. That the sediment contribution at all time in history have influence on current sediment contribution. The heavy tail of the sediment contribution curve is more evident with the order of the time Hausdorff fractal derivative α is smaller.

The turbulent diffusion increasing is caused with of the order of the space Hausdorff fractal derivative β decreasing. It illustrates that the space Hausdorff fractal derivative advection-dispersion equation can describe the vertical distribution of sediment concentration caused by the fast displacement of turbulent super-diffusion. In fig. 1(b), with the order of space Hausdorff fractal β decreasing, the diffusion rate of sediment change faster than normal diffusion. Since the turbulent diffusion increasing with the order of the space fractal derivative β decreasing. It can be explained that the smaller space fractal derivative β, the stronger turbulent diffusion, and it exhibits super-diffusive behavior.

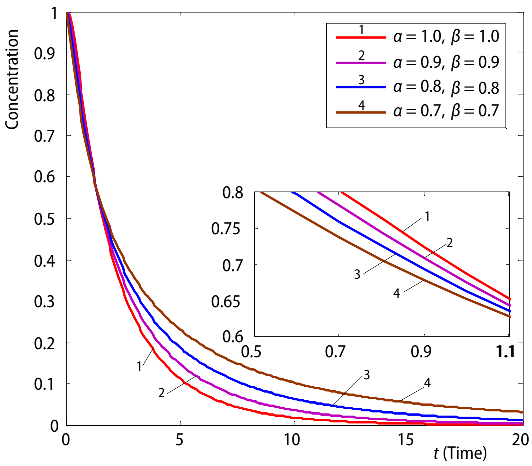

Figure 2. Dimensionless numerical simulation results of time-space Hausdorff fractal derivative advection-dispersion model, diffusion coefficient ε_{sy} = 1.0, the settling velocity ω = 1.0, y = 10

As shown in fig. 2, in the time-space fractal Hausdorff derivative model, anomalous behavior in space is obvious in early time, the diffusion is faster than normal diffusion with the space fractal derivative order β decreasing, which displays the super-diffusion in the initial interval $t \in (0,1.5]$. But $t \in (2,20]$, the memory effect is obvious as time increasing, the diffusion is slower than normal diffusion with the time fractal derivative decreasing, which exhibits the sub-diffusion. Therefore, in the time-space Hausdorff fractal derivative advection-dispersion model, the anomalous diffusion is accelerated or slowed down, it needs for specific practical problems.

To investigate the efficiency of the present the fractal Hausdorff derivative advection-dispersion model, we compare its sediment profiles with the experimental data. In this study, the numerical results are compared with the experimental data from the Laboratory of Institute of Hydraulic Engineering and Water Resources Management, in their experiments, the mean concentration of sediment are 3 g per liter and 20 g per liter. The more data we could collect, the more accurate the parameters of fractal derivative model will be. Because the experiment data are not enough to obtain sufficiently accurate parameters of the Hausdorff fractal derivative model, we compare the Hausdorff fractal derivative model with experimental data, which *best fits* the experimental profiles to obtain the parameters of fractal derivative model. Figure 3 shows the best-fit results with experimental data. Figure uses the data from the experiments to calibrate the value of the Hausdorff fractal derivative orders α and β. As shown in fig. 3, the Hausdorff fractal derivative model gives a better agreement with experimental data except for 3 g per liter in $t \in (900,1500]$ the profile of suspended sediment. Maybe the inconsistency is caused by the error of measured data. In fig. 3, we can obtain the space derivative value of 20 g per liter is smaller than 3 g per liter, because the fast displacement is strong com-

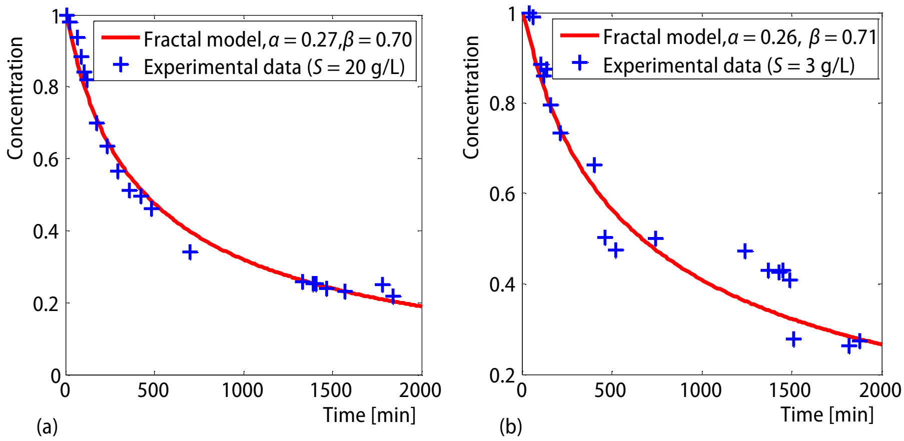

Figure 3. Numerical results of the Hausdorff fractal derivative model in describing the experimental data; [11] (a) S = 20 g/L, D =1.4 cm$^\beta$/min$^\alpha$, ω = 1.5 cm/min$^\alpha$, y = 19.5 cm, (b) S = 3 g/L, D = 1.2 cm$^\beta$/min$^\alpha$, ω = 1.5 cm/min$^\alpha$, y = 19.5 cm

pare with history-dependency in high concentration, it exhibits super-diffusion processes. In contrast, the time derivative value of 3 g per liter is smaller than 20 g per liter, because the fast displacement is less than history-dependency in low concentration, it exhibits sub-diffusion processes. So it indicates that the Hausdorff fractal derivative advection-dispersion model can reliably describe the sediment transport in experiments, via fitting parameters α and β.

In summary, Model (3) captures the trapping effect of suspended sediment transport using the time Hausdorff fractal derivative term with an index α, and characterizes the fast displacement of the suspended sediment using the space Hausdorff fractal derivative term with an index β. By adjusting the two indices in the Hausdorff fractal derivative advection-dispersion model, we can conveniently capture various anomalous behaviors for suspended sediment transport in unsteady flow.

Conclusion

This study makes an attempt to develop a Hausdorff fractal derivative advection-dispersion model to characterize the suspended sediment transport. An approximate solution is obtained to simulate the suspended sediment transport in unsteady flow. According to the comparison results, the Hausdorff fractal derivative model can well capture the main features of anomalous sediment transport and give a good agreement with the experimental data. Time and space Hausdorff fractal derivative orders α and β are two key parameters to characterize the history-dependency and fast displacement of the suspended sediment transport in unsteady flow. However, the advantages and application potentials of the Hausdorff fractal derivative model should be further investigated through more theoretical discussions and experimental verification.

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