

## ON NUMERICAL SOLUTIONS FOR THE CAPUTO-FABRIZIO FRACTIONAL HEAT-LIKE EQUATION

by

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*In this article, Laplace homotopy analysis method in order to solve fractional heat-like equation with variable coefficients, are introduced. Laplace homotopy analysis method, founded on combination of homotopy methods and Laplace transform is used to supply a new analytical approximated solutions of the fractional partial differential equations in case of the Caputo-Fabrizio. The solutions obtained are compared with exact solutions of these equations. Reliability of the method is given with graphical consequens and series solutions. The results show that the method is a powerfull and efficient for solving the fractional heat-like equations with variable coefficients.*

Key words: *Laplace homotopy analysis method, fractional heat-like equations, Caputo-Fabrizio derivative, approximate solution*

### Introduction

In the last few years, respectable concern in fractional calculus applied in numerous studies, such as regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, viscoelasticity, electrical circuits, electro-analytical chemistry, biology, control theory, etc. [1-4]. Besides there has been a significant theoretical development in fractional differential equations and its applications [5-10]. On the other hand, fractional derivatives supply an important implement for the definition of hereditary characteristics of different necessities and treatment. Some scientists have been interested in improving new definition of fractional derivative. These derivative definitions change from Riemann-Liouville derivative to the Caputo-Fabrizio derivative introduced by Caputo and Fabrizio [11-19]. They are claimed that the new derivative has interesting properties than the former derivatives. Their derivative does not run into an any singularity, thus a new fractional order derivative without a singular kernel can efficiently describe the effect of memory and also able to portray material heterogeneities and structures in different cases, which are physically symbolized by distinction or variation of the average.

In this paper, we apply the Laplace homotopy analysis method (LHAM) to find analytical approximated solution for fractional heat and wave like equations using in case of every two fractional operators. The LHAM is a combining of the semi analytical method projected by Liao and the Laplace transform [20, 21]. Some writers have projected various systems for fractional PDE with every two fractional operators. In [22], Dehghan practiced the HAM to

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solve fractional PDE with in case of Liouville-Caputo. In [23], is studied a fractional differential equation with a changeable coefficient. Jafari *et al.* [24] applied the HAM in order to solve the high orderly fractional differential equation analyzed by Diethelmand and Ford [25]. In [26], is produced a mathematical analysis of an example studied the Caputo-Fabrizio fractional derivative, where analytical and calculation advances are fined. Morales-Delgado *et al.* [27] presented LHAM to supply new solutions in case of every two fractional operators. Other analytical advances that could be of concern are introduced in [28-31].

In this work, we think the following the 3-D fractional heat- and wave-like equations with the initial conditions of the shape [32]:

$$\frac{\partial^\alpha u(x, y, z, t)}{\partial t^\alpha} = f \frac{\partial^2 u(x, y, z, t)}{\partial x^2} + g \frac{\partial^2 u(x, y, z, t)}{\partial y^2} + h \frac{\partial^2 u(x, y, z, t)}{\partial z^2} \quad (1)$$

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < c, \quad t > 0, \quad 0 < \alpha \leq 2$$

$$u(x, y, z, 0) = \Xi(x, y, z), \quad u_t(x, y, z, 0) = \Theta(x, y, z) \quad (2)$$

where  $\alpha$  is a parameter defining the fractional derivative and  $u_t$  is the value of variety of temperature at a point over time. The  $u = u(x, y, z, t)$  is temperature as a function of time and space, while  $u_{xx}$ ,  $u_{yy}$ , and  $u_{zz}$  are the second spatial derivatives (caloric conductions) of temperature in  $x$ -,  $y$ -, and  $z$ -directions, respectively. In addition to,  $f$ ,  $g$ , and  $h$  are any functions in  $x$ ,  $y$ , and  $z$ .

In case  $0 < \alpha < 1$ , eq. (1) give the fractional heat-like equation with variable coefficients. And in case  $1 < \alpha < 2$ , eq. (1) give the fractional wave-like equation which styles abnormal diffusive and subdiffusive systems, definition of fractional casual walk, unification of diffusion and wave propagation process [33-36]. Recently, in [37], eq. (1) was applied to types in some domains like fluid mechanics. In [38], scientists applied variational iteration method in order to find approximation solutions of 1-D of eq. (1). In [32, 38, 39], writers studied the multi-dimensional time fractional heat-like equations by using LHAM.

### Basic definitions of fractional calculus theory

We first illustrate the main descriptions and various features of the fractional calculus theory [2] in this section.

*Definition 2.1.* The Riemann-Liouville fractional integral operator of order  $\alpha$  ( $\alpha \geq 0$ ) is defined:

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad \alpha > 0, \quad x > 0 \quad (3)$$

$$J^0 u(x) = u(x)$$

*Definition 2.2.* The Caputo fractional derivatives of order  $\alpha$  is defined:

$${}_0^C D_t^\alpha u(t) = J^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-z)^{m-\alpha-1} \frac{d^m}{dz^m} u(z) dz, \quad (4)$$

$$m-1 < \alpha \leq m, \quad t > 0$$

where  $D^m$  is the classical differential operator of order  $m$ .

For the Caputo derivative we have:

$$D^\alpha t^\beta = 0, \quad \beta < \alpha, \quad D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \quad \beta \geq \alpha \quad (5)$$

The efficacy of this definition is confined to functions  $u$  such that  $u^{(m)} \in L_1(a, b)$ .

If  $u^{(m)} \in L_1(\mathbb{R}^+)$  and if  $u^{(m)}(t)$  is of exponential order  $v_m$ , with  $v_m > 0, \forall m = 0, 1, 2, \dots, n - 1$ , the form advised in the sources [11]:

$$L[{}_0^C D_x^\alpha u](z) dt = \frac{1}{z^{m-\alpha}} \{ z^m L[u(x, t)](z) - z^{m-1} u(x, 0) - \dots - u^{(m-1)}(x, 0) \} \quad (6)$$

for  $Re(z) > l, l = \max\{v_m : m = 0, 1, 2, \dots, n - 1\}$ .

Then,

$$L[{}_0^C D_t^\alpha u(x, t)](z) = z^\alpha L[u(x, t)](z) - z^{\alpha-1} u(x, 0), \quad 0 < \alpha \leq 1$$

$$L[{}_0^C D_t^\alpha u(x, t)](z) = z^\alpha L[u(x, t)](z) - z^{\alpha-1} u(x, 0) - u'(x, 0), \quad 1 < \alpha \leq 2$$

Therefore, in eq. (4) if transformations happen:

$$(t - z)^{m-\alpha-1} \rightarrow \exp \frac{-\alpha(t-z)}{1-\alpha} \quad \text{and} \quad \frac{1}{\Gamma(m-\alpha)} \rightarrow \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$$

the new definition of fractional operator is expressed by Caputo and Fabrizio [11, 31].

*Definition 2.3.* Let  $u \in H^1(a, b)$ , the new fractional Caputo derivative is defined:

$${}_0^{CF} D_t^\alpha [u(t)] = \frac{(2-\alpha)M(\alpha)}{1-\alpha} \int_0^t u^{(n)}(z) \exp \left[ -\alpha \frac{t-z}{1-\alpha} \right] dz, \quad b > a, \quad \alpha \in (0, 1] \quad (7)$$

The  $M(\alpha)$  is a standardization function that  $M(0) = M(1) = 1$  [11]. Then eq. (7) does not have singularities at  $t = z$ .

But if  $u \notin H^1(a, b)$ , eq. (7) can be re-written:

$${}_0^{CF} D_t^\alpha [u(t)] = \frac{\alpha M(\alpha)}{1-\alpha} \int_0^t [u(t) - u(z)] \exp \left[ -\alpha \frac{t-z}{1-\alpha} \right] dz \quad (8)$$

*Definition 2.4.* The fractional integral of order  $\alpha$  of  $u$  is defined:

$$I_t^\alpha [u(t)] = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(z) dz, \quad t \geq 0 \quad (9)$$

where  $0 < \alpha < 1$ .

*Remark [12].* According to the *Definition 2.4*, the fractional integral of Caputo type of function of order  $0 < \alpha < 1$  is a medial between function  $u$  and its integral of order one.

Thus

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1$$

The previous formulation gives an expressed for:

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 \leq \alpha \leq 1$$

Therefore, in [12] is rewrote the recent fractional Caputo derivative:

$$D_t^\alpha [u(t)] = \frac{1}{1-\alpha} \int_0^t u'(z) \exp \left[ -\alpha \frac{t-z}{1-\alpha} \right] dz, \quad 0 < \alpha < 1 \quad (10)$$

*Theorem 1.* (see [11,12] for proof) If the function  $u(z)$  as:

$$u^i(a) = 0, \quad i = 1, 2, \dots, n$$

in the new fractional Caputo derivative, therefore, we write:

$$D_t^\alpha \{D_t^n [u(z)]\} = D_t^n \{D_t^\alpha [u(z)]\}$$

*Definition 2.5.* After *Definition (2.3)*, if  $\alpha \in (0,1]$  and  $n \in \mathbb{N}$ , we can define the Laplace transform in case of C-F [11, 32]:

$$\begin{aligned} L_{[0]}^{\text{CF}} D_t^{(\alpha+1)} u(t)(z) &= \frac{1}{1-\alpha} L[u^{(\alpha+n)}(t)] \exp\left(-\frac{\alpha}{\alpha-1} t\right) = \\ &= \frac{z^{n+1} L[u(t)] - z^n u(0) - z^{n-1} u'(0) - \dots - u^{(n)}(0)}{z + \alpha(1-z)} \end{aligned} \quad (11)$$

From eq. (11),

$$L_{[0]}^{\text{CF}} D_t^\alpha u(t)(z) = \frac{zL[u(t)] - u(0)}{z + \alpha(1-z)}, \quad n = 0 \quad (12)$$

$$L_{[0]}^{\text{CF}} D_t^{(\alpha+1)} u(t)(z) = \frac{z^2 L[u(t)] - su(0) - u'(0)}{z + \alpha(1-z)}, \quad n = 1 \quad (13)$$

### The LHAM for fractional heat- and wave-like equations with the operator of Caputo-Fabrizio

$$\begin{aligned} {}_0^{\text{CF}} D_t^\alpha u(x, y, z, t) - f \frac{\partial^2 u(x, y, z, t)}{\partial x^2} - g \frac{\partial^2 u(x, y, z, t)}{\partial y^2} - h \frac{\partial^2 u(x, y, z, t)}{\partial z^2} = 0 \end{aligned} \quad (14)$$

$$m-1 < \alpha + n \leq m$$

the initial conditions are:

$$\frac{\partial^i u(x, y, z, 0)}{\partial t^i} = u_i(x, y, z), \quad i = 0, 1, \dots, m-1 \quad (15)$$

and

$$u(0, y, z, t) = \varepsilon_0(t), \quad u(1, y, z, t) = \varepsilon_1(t), \quad t \geq 0$$

in case of the Caputo-Fabrizio fractional derivative, we can write:

$$\begin{aligned} L_{[0]}^{\text{CF}} D_t^{(\alpha+n)} u(x, y, z, t)(s) &= \frac{1}{s + \alpha(1-s)} \left\{ s^{n+1} L[u(x, y, z, t)] - \right. \\ &\quad \left. - s^n u(x, y, z, 0) - s^{n-1} u'(x, y, z, 0) - \dots - u^{(n)}(x, y, z, 0) \right\} \end{aligned} \quad (16)$$

where  $s > 0$ ,  $L[u(x, y, z, t)](\zeta) = \Psi(x, y, z, \zeta)$ , then eq. (16):

$$\begin{aligned} \Psi(x, y, z, \zeta) &= -p \frac{\zeta + \alpha(1-\zeta)}{\zeta^{n+1}} \left( -f \frac{\partial^2}{\partial x^2} - g \frac{\partial^2}{\partial y^2} - h \frac{\partial^2}{\partial z^2} \right) \Psi(x, y, z, \zeta) + \\ &\quad + \frac{1}{\zeta^{n+1}} [\zeta^n u_0(x, y, z) + \zeta^{n-1} u_1(x, y, z) + \dots + u_n(x, y, z)] \end{aligned} \quad (17)$$

in eq. (17),  $\Psi(x, y, z, \zeta) = L[u(x, y, z, t)]$  and

$$\Psi(0, y, z, \zeta) = L[\varepsilon_0(y, z, t)], \quad \Phi(1, y, z, \zeta) = L[\varepsilon_1(y, z, t)], \quad z \geq 0$$

Then

$$\Psi(x, y, z, \zeta) = \sum_{i=0}^{\infty} p^i \Psi_i(x, y, z, \zeta), \quad i = 0, 1, 2, \dots, \quad (18)$$

eq. (18) is the solution of eq. (17).

Substituting eq. (18) into eq. (17), we obtain:

$$\begin{aligned} \sum_{i=0}^{\infty} p^i \Psi_i(x, y, z, \zeta) = & -p \frac{\zeta + \alpha(1-\zeta)}{\zeta^{n+1}} \left( -f \frac{\partial^2}{\partial x^2} - g \frac{\partial^2}{\partial y^2} - h \frac{\partial^2}{\partial z^2} \right) \sum_{i=0}^{\infty} p^i \Psi_i(x, y, z, \zeta) + \\ & + \frac{1}{\zeta^{n+1}} [\zeta^n u_0(x, y, z) + \zeta^{n-1} u_1(x, y, z) + \dots + u_n(x, y, z)] \end{aligned} \quad (19)$$

from the coefficients of powers of  $p$ :

$$\begin{aligned} p^0 : \Psi_0(x, y, z, \zeta) &= \frac{1}{\zeta^{n+1}} [\zeta^n u_0(x, y, z) + \zeta^{n-1} u_1(x, y, z) + \dots + u_n(x, y, z)] \\ p^1 : \Psi_1(x, y, z, \zeta) &= \frac{\zeta + \alpha(1-\zeta)}{\zeta^{n+1}} \left( f \frac{\partial^2}{\partial x^2} + g \frac{\partial^2}{\partial y^2} + h \frac{\partial^2}{\partial z^2} \right) \Psi_0(x, y, z, \zeta) \\ p^2 : \Psi_2(x, y, z, \zeta) &= \frac{\zeta + \alpha(1-\zeta)}{\zeta^{n+1}} \left( f \frac{\partial^2}{\partial x^2} + g \frac{\partial^2}{\partial y^2} + h \frac{\partial^2}{\partial z^2} \right) \Psi_1(x, y, z, \zeta) \\ &\vdots \\ p^{n+1} : \Psi_{n+1}(x, y, z, \zeta) &= \frac{\zeta + \alpha(1-\zeta)}{\zeta^{n+1}} \left( f \frac{\partial^2}{\partial x^2} + g \frac{\partial^2}{\partial y^2} + h \frac{\partial^2}{\partial z^2} \right) \Psi_n(x, y, z, \zeta) \end{aligned}$$

and eq. (19) yields the approximate solution of eq. (17):

$$H_n(x, y, z, \zeta) = \sum_{i=0}^{\infty} \Psi_i(x, y, z, \zeta) \quad (20)$$

if we apply the inverse of the Laplace transform of eq. (20), we can write solution of eq. (14):

$$u_{\text{approx}}(x, y, z, t) \approx L^{-1}[H_n(x, y, z, \zeta)] \quad (21)$$

Let us define:

$$S_n(x, y, z, t) = L^{-1}\left[\sum_{i=0}^n \Psi_i(x, y, z, \zeta)\right],$$

where  $S_n(x, y, z, t)$  is the  $n^{\text{th}}$  partial sum of the infinite series of approximate solution [31], then the relative error,  $RE(\%)$ , is calculated:

$$RE(\%) = \left| \frac{S_n(x, y, z, t) - u_{\text{exact}}(x, y, z, t)}{u_{\text{exact}}(x, y, z, t)} \right| \cdot 100$$

### The 2-D fractional heat-like equation

In this part, fractional heat-like equations are solved using the Caputo-Fabrizio fractional operators in order to demonstrate the effectiveness of the LHAM, in addition the convergence and stability of the method are discussed.

Consider the 2-D fractional heat-like equation in case of the Caputo-Fabrizio:

$${}^c_0D_t^\alpha u(x, y, t) - \frac{\partial^2 u(x, y, t)}{\partial x^2} - \frac{\partial^2 u(x, y, t)}{\partial y^2} = 0, \quad (22)$$

$$0 < x, \quad y < 2\pi, \quad t > 0, \quad 0 < \alpha \leq 1$$

by the initial condition:

$$u(x, y, 0) = \sin(x)\sin(y) \quad (23)$$

Now, using the LHAM, we have:

$$p^0 : \Psi_0(x, y, \zeta) = \frac{\sin(x)\sin(y)}{\zeta}$$

$$p^1 : \Psi_1(x, y, \zeta) = -\frac{\zeta + \alpha(1-\zeta)}{\zeta^2} \left( -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \Psi_0(x, y, \zeta) =$$

$$= -\frac{2[\zeta + \alpha(1-\zeta)]\sin(x)\sin(y)}{\zeta^3}$$

$$p^2 : \Psi_2(x, y, \zeta) = -\frac{\zeta + \alpha(1-\zeta)}{\zeta^2} \left( -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \Psi_1(x, y, \zeta) =$$

$$= \frac{4[\zeta + \alpha(1-\zeta)]^2 \sin(x)\sin(y)}{\zeta^5}$$

$$\vdots$$

the approximate solution is:

$$H_n(x, y, \zeta) = \sum_{i=0}^n \Psi_i(x, y, \zeta) = \sin(x)\sin(y) \sum_{m=0}^n \frac{\{-2[\zeta + \alpha(1-\zeta)]\}^m}{\zeta^{2m+1}} \quad (24)$$

we have:

$$u_3(x, y, t) = L^{-1}[H_3(x, y, \zeta)] = H_3(x, y, t) = \sin(x)\sin(y) \left[ 1 + 2t(-1 + \alpha) - \right.$$

$$\left. -\frac{t^6 \alpha^3}{90} + t^2(2 - 5\alpha + 2\alpha^2) + \frac{4}{3}t^3(-1 + 4\alpha - 4\alpha^2 + \alpha^3) + \right.$$

$$\left. + \frac{1}{5}t^5(-\alpha^2 + \alpha^3) - \frac{1}{6}t^4(6\alpha - 13\alpha^2 + 6\alpha^3) \right] \quad (25)$$

The result given in figs. 1-3, they shows that our approximate solutions are in good agreement with the exact values.

Figure 1. The surface graph of exact solution of the 2-D fractional heat-like equation ( $y = 2\pi/3$ )

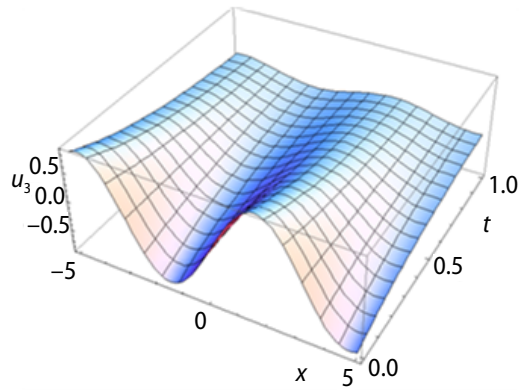


Figure 2. The surface graph of  $u_3(x, y, t)$  approximate solutions of the 2-D fractional heat-like equation in case of the Caputo-Fabrizio ( $y = 2\pi/3$ );  
 (a)  $u_3(x, y, t)$  when  $\alpha = 0.01$ ,  
 (b)  $u_3(x, y, t)$  when  $\alpha = 0.1$ ,  
 (c)  $u_3(x, y, t)$  when  $\alpha = 0.5$ ,  
 (d)  $u_3(x, y, t)$  when  $\alpha = 0.9$

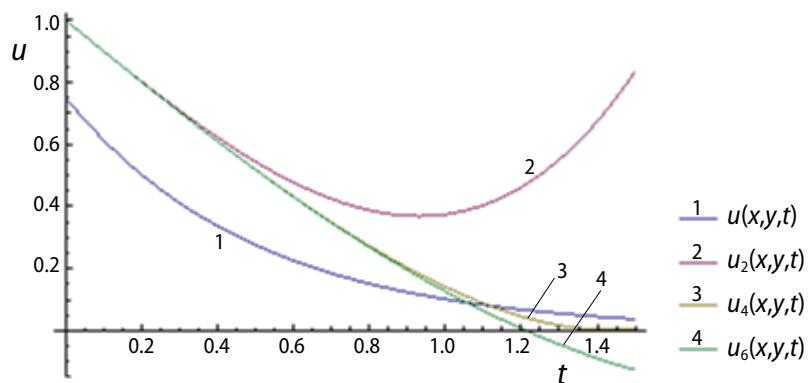
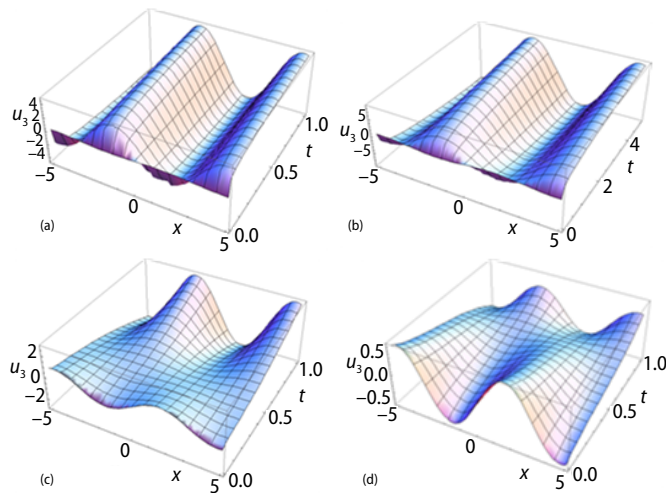


Figure 3. The  $u_i(x, t)$  solutions of the 2-D fractional heat-like equation in case of the Caputo-Fabrizio when  $k = 2, 4, 6$  and its exact solution ( $\alpha = 0.5$ ,  $x = y = 2\pi/3$ ).

## Conclusions

In this study the LHAM has utilized in order to find approximate analytical solution of 2-D fractional heat-like equation in case of the Caputo-Fabrizio. We have compared the approximate solutions received in the sight of LHAM with those outcomes received from the exact analytical solutions. This operation indicates an accurate understanding between the LHAM and exact outcomes. From the outcomes, it is clear that the LHAM yields very accurate and convergent approximate solutions using only a few iterates in fractional problems. Because the Laplace transform permits one in many positions to get over the deficiency chiefly produced by unsatisfied boundary or initial conditions, the LHAM is a strong new method which requires inferior calculation time and is much easier and more useful than the HAM. The work emphasized our belief that the present method can be applied as an alternative to get approximate analytic solutions of different kinds of fractional linear and non-linear PDE in Caputo-Fabrizio fractional derivative sense applied in mathematics, physics and engineering.

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