FRACTIONAL VARIATIONAL ITERATION METHOD FOR TIME-FRACTIONAL NON-LINEAR FUNCTIONAL PARTIAL DIFFERENTIAL EQUATION HAVING PROPORTIONAL DELAYS

by

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In this paper, time-fractional non-linear partial differential equation with proportional delays are solved by fractional variational iteration method taking into account modified Riemann-Liouville fractional derivative. The numerical solutions which are calculated by using this method are better than those obtained by homotopy perturbation method and differential transform method with same data set and approximation order. On the other hand, to improve the solutions obtained by fractional variational iteration method, residual error function is used. With this additional process, the resulting approximate solutions are getting closer to the exact solutions. The results obtained by taking into account different values of variables in the domain are supported by compared tables and graphics in detail.

Key words: *modified Riemann-Liouville derivative, proportional delays, time-fractional pde, fractional variational iteration method*

Introduction

Fractional differential equations become a fundamental tool to understand real life problems and it is used at almost all disciplines. There are various studies on fractional differential equations [1]. Some of these equations are formed by replacing the positive integer order derivatives with modified fractional derivatives, so it is aimed to find out what the behavior is. In order to determine the solutions that guide the behavioral state, those are solved numerically. The most realistic models of ODE do not have analytic solutions so that the numerical and approximation methods should be used in order to solve such problems [2]. The variational iteration method (VIM) is also applied successfully to both numerous linear and non-linear fractional order problems by many authors [3-5]. Besides that, there are also several methods used in order to solve the non-linear problems. But, most of the authors claimed that VIM and the numerical results demonstrate that the VIM is relatively accurate and also easily implemented method such as Adomian decomposition method, differential transform method (DTM) and some other methods [6, 7]. On the other hand, fractional PDE that appear in many physical phenomena are also studied by some researchers using some treated types of VIM [8-13]. Partial functional-differential equations with proportional delays represent a particular class of delay PDE and also these are solved by several authors such as [14, 15].

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On the other hand Polyanin and Zhurov [16] suggested a method for constructing exact solutions to non-linear delay reaction-diffusion equations. Additionally Abazari and Ganji [17] proposed 2-D DTM and its reduced form to obtain the solution of PDE with proportional delay. Then Sakar *et al.* [18] proposed homotopy perturbation method (HPM) for numerical solutions of these kinds of special equations.

In this paper, we examine non-linear fractional PDE which have proportional delays. Previously, Ghaneai *et al.* [19] applied modified VIM to non-linear PDE, Abazari and Ganji [17] studied these kind of fractional equations by using extended DTM and Sakar *et al.* [18] applied HPM taking into account Caputo derivative definition to aforementioned equations. Very recently, Singh and Kumar [20] has just proposed to use an alternative VIM considering Caputo sense derivative. The differences of our study among previous studies are considering modified Riemann-Liouville derivative operator [21] firstly, improvement of the solutions with residual error function [22] secondly, so that having approximately at least 10^{-4} times more accurate data, and finally obtaining semi-analytic solutions, that is, approximate solutions are functions of x and t.

Now, let us consider following time-fractional PDE with proportional delays of the general form:

$$\mathrm{D}_t^\alpha u(x,t) = f\left[x,t,u(p_0x,q_0t),\mathrm{D}_x u(p_1x,q_1t),\ldots,\mathrm{D}_x^n u(p_nx,q_nt)\right] \quad n=0,1,2,\ldots \tag{1}$$

subject to the initial conditions $u^{(k)}(x,0)=\eta_k(x)$ for $k=0,1,\ldots,m$, $m<\alpha\le m+1$, and $m\in\mathbb{N}$ where $(x,t)\in[0,1]\times[0,1]$, $\eta_k(x)$ is a specified initial function, p_i, $q_j\in(0,1)$ for $i,j\in\mathbb{N}$, α is a parameter describing the order of the time fractional derivative and $u(x,t)$ is the exact solution. For fractional integrals, the Riemann-Liouville fractional integral definition and for fractional derivatives modified fractional derivative definition [21, 23, 24] are used in our approach. This definition is just a modification on the definition of Riemann-Liouville derivative and it is strictly equivalent to the Caputo via Riemann fractional derivative. The definitions used are briefly introduced:

Definition. Riemann-Liouville fractional integral of a continuous function $f: R^2 \to R$, $(x,t)\to f(x,t)$ with respect to t of order α is:

$${}^{RL}I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x,\tau)\,\mathrm{d}\tau \qquad \alpha>0 \tag{2}$$

Definition. The modified Riemann-Liouville fractional derivative is:

$${}_0^{mRL}\mathrm{D}_t^\alpha f(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\mathrm{D}_t\left\{\displaystyle\int_0^t (t-\tau)^{-\alpha}\left[f(x,\tau)-f(x,0)\right]\mathrm{d}\tau\right\}, & \text{for } 0<\alpha<1, \\ {}^{mRL}\mathrm{D}_t^{\alpha-n}[\mathrm{D}_t^n f(x,t)], & \text{for } \mathrm{n}\le\alpha<n+1,\ n\ge 1,\ n\in\mathbb{N} \end{cases} \tag{3}$$

where D_t^n denotes the n^{th} partial derivative with respect to t, while ${}^{mRL}\mathrm{D}_t^\alpha$ is modified Riemann Liouville derivative of order α [21].

Furthermore, modified Riemann-Liouville fractional derivative is strictly equivalent to the Grunwald-Letnikov fractional derivative [25] and has valuable advantages according to both standard Riemann-Liouville and Caputo fractional derivatives. For instance, it is defined for arbitrary continuous (can also be non-differentiable) functions and the fractional derivative of a constant is equal to zero. If the function is not defined at the origin, the fractional derivative will not exist. In order to overcome this manner Atangana and Secer [26] proposed to take finite part of the fractional derivative order operator which is based on the concept of finite part approach of Estrada and Kanwal [27].

With this definition some important properties can be introduced:

– fractional integration of a fractional derivative

$$ {}_0^{RL}I_t^{\alpha}\ {}_0^{mRL}\mathrm{D}_t^{\alpha} f(x,t) = f(x,t) - f(x,0), \quad 0 < \alpha \le 1 \tag{4}$$

– local integration

$$ {}_0^{RL}I_t^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x,\tau)\,\mathrm{d}\tau = \frac{1}{\Gamma(\alpha+1)}\int_0^t f(x,\tau)(\mathrm{d}\tau)^{\alpha} \tag{5}$$

where fractional derivative of compounded function is defined:

$$ (\mathrm{d}\tau)^{\alpha} \cong \Gamma(1+\alpha)\,\mathrm{d}\tau, \quad 0 < \alpha < 1 \tag{6}$$

in the view of [21, 23, 24].

For comparison purposes, especially, we take the derivative order $\alpha \in (0,1]$ so the problem that we have, now becomes:

$$ \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = f\left[x, t, u(p_0 x, q_0 t), \mathrm{D}_x u(p_1 x, q_1 t), \ldots, \mathrm{D}_x^n u(p_m x, q_m t)\right] \quad m = 0,1,2,\ldots \tag{7}$$

subject to the initial condition $u(x,0) = \eta(x)$.

Process of fractional VIM (FVIM) for fractional PDE

According to standard VIM theory which was firstly proposed by He [28], we shall regenerate a corrected functional that allows us to construct an iteration formula in order to find fixed point of that formula. Based on this structure, the FVIM has already been presented and used by many authors [29-32].

We are dealing with problem of eq. (7), so its corrected functional is written as in the form of:

$$ u_{n+1}(x,t) = u_n(x,t) + $$
$$ + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(t,\tau)\left\{ {}_0^{mRL}\mathrm{D}_\tau^{\alpha} u_n(x,\tau) - \tilde{f}\left[x,\tau,\tilde{u}(p_0 x, q_0 \tau), \mathrm{D}_x \tilde{u}_n(p_1 x, q_1 \tau), \ldots\right]\right\}(\mathrm{d}\tau)^{\alpha} \tag{8}$$

where $\lambda(t,\tau)$, Lagrange multiplier, can be identified optimally via variational theory in which $\tilde{f}$ is restricted variation, so is $\delta\tilde{f} = 0$ consequently. Making eq. (8) stationary yields following conditions:

$$ \lambda(t,\tau) + 1 = 0, \quad {}_0^{mRL}\mathrm{D}_\tau^{\alpha}\lambda(t,\tau) = 0 \tag{9}$$

and it is easily understood that the trivial solution of those system is $\lambda(t,\tau) = -1$. Now our iteration formula is:

$$ u_{n+1}(x,t) = u_n(x,t) - $$
$$ - \frac{1}{\Gamma(\alpha+1)}\int_0^t \left\{ {}_0^{mRL}\mathrm{D}_\tau^{\alpha} u_n(x,\tau) - f\left[x,\tau,u(p_0 x, q_0 \tau), \mathrm{D}_x u_n(p_1 x, q_1 \tau), \ldots\right]\right\}(\mathrm{d}\tau)^{\alpha} \tag{10}$$

which will have a fixed point $u(x,t)$ taking into account a special initial approximate function $u_0(x,t)$ that can be freely chosen if it satisfies the initial and boundary conditions of the problem. Approximate solutions are determined:

$$u_n(x,t), n \in \mathbb{N} \quad \text{where} \quad \lim_{n\to\infty} u_n(x,t) = u(x,t) \tag{11}$$

Improvement of solutions obtained by FVIM

Improvement with residual error function

In case the exact solution of the problem is not known or can not be obtained analytically, in order to check the sensitivity of approximate solutions obtained with FVIM, we will use the residual error function which allows us to approach the desired solution $u(x,t)$ as $u_n(x,t)+e_n(x,t)$. We will correct the approximated solution $u_n(x,t)$ using the residual error function $e_n(x,t)$.

Assume that the n^{th} order approximate solution $u_n(x,t)$ satisfies:

$${}_0^{mRL}\mathrm{D}_t^{\alpha} u_n(x,t) - \overline{f}\left[u_n(p_0x,q_0t), \mathrm{D}_x u_n(p_1x,q_1t),..., \mathrm{D}_x^n u_n(p_mx,q_mt)\right] = g(x,t) + R(x,t) \tag{12}$$

such that a residual function remains as $R(x,t)$ on the right hand side of eq. (12) where $g(x,t)$ is non-homogenous function removed from f. Since $u(x,t)$ is the exact solution of eq. (7), eq. (12) can be also written:

$${}_0^{mRL}\mathrm{D}_t^{\alpha} u(x,t) - \overline{f}\left[u(p_0x,q_0t), \mathrm{D}_x u(p_1x,q_1t),..., \mathrm{D}_x^n u(p_mx,q_mt)\right] = g(x,t) \tag{13}$$

Subtracting eq. (13) from eq. (12) yields:

$$\begin{aligned} &{}_0^{mRL}\mathrm{D}_t^{\alpha}\left[u(x,t)-u_n(x,t)\right] - \\ &-\Big\{\overline{f}\left[u(p_0x,q_0t), \mathrm{D}_x u(p_1x,q_1t),..., \mathrm{D}_x^n u(p_mx,q_mt)\right] - \\ &-\overline{f}\left[u_n(p_0x,q_0t), \mathrm{D}_x u_n(p_1x,q_1t),..., \mathrm{D}_x^n u_n(p_mx,q_mt)\right]\Big\} = -R(x,t) \end{aligned} \tag{14}$$

Denoting by $e_n(x,t)$ the residual error function of $u_n(x,t)$ and taking in consideration that ${}_0^{mRL}\mathrm{D}_t$ is a linear operator we have the error differential equation with homogenous initial condition:

$$\begin{aligned} &{}_0^{mRL}\mathrm{D}_t^{\alpha}\left[e_n(x,t)\right] - \\ &-\Big\{\overline{f}\left[(u_n+e_n)(p_0x,q_0t), \mathrm{D}_x(u_n+e_n)(p_1x,q_1t),..., \mathrm{D}_x^n(u_n+e_n)(p_mx,q_mt)\right] - \\ &-\overline{f}\left[u_n(p_0x,q_0t), \mathrm{D}_x u_n(p_1x,q_1t),..., \mathrm{D}_x^n u_n(p_mx,q_mt)\right]\Big\} = -R(x,t) \end{aligned} \tag{15}$$

subject to $e_n(x,0)=0$.

Solving this by a numerical method, such as FVIM, $e_n(x,t)$ is found numerically, therefore, the solution $u_n(x,t)$ is improved by adding that term.

Numerical experiments

In this section, time-fractional PDE with proportional delays of eq. (7) that were solved by using DTM [17] and earlier by HPM [18], will be considered.

Example 1

The first time fractional PDE is:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathrm{D}_{xx}u(x,t) + \mathrm{D}_x u\left(x,\frac{t}{2}\right)u\left(\frac{x}{2},\frac{t}{2}\right) + \frac{1}{2}u(x,t) \quad 0<\alpha\le 1 \tag{16}$$

with initial condition $u(x,0)=x$, $(x,t)\in[0,1]\times[0,1]$ and the exact solution is $u(x,t)=x e^t$ when $\alpha=1$, which was already solved by DTM in [17], HPM in [18]. The numerical data are calculated separately for four cases in accordance with the value of α as follows:

– Case $\alpha=1$

So the equation becomes first order PDE with respect to t and the iteration formula for FVIM is constructed:

$$u_{n+1}(x,t) = u_n(x,t) -$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_0^t\left[{}^{mRL}_{\ 0}\mathrm{D}^\alpha_\tau u_n(x,\tau) - \mathrm{D}_{xx}u_n(x,\tau) + \mathrm{D}_x u_n\left(x,\frac{\tau}{2}\right)u_n\left(\frac{x}{2},\frac{\tau}{2}\right) + \frac{1}{2}u_n(x,\tau)\right](\mathrm{d}\tau)^\alpha \tag{17}$$

which will have a fixed point $u(x,t)$ taking initial approximate function $u_0(x,t)=x$. While n is increasing, approximate solutions of order n are indicated:

$$u_1(x,t) = x\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}\right] \tag{18}$$

$$u_2(x,t) = x\left\{1+\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(2^{1-\alpha}+1)t^{2\alpha}}{2\Gamma(2\alpha+1)} + \frac{\Gamma(2\alpha+1)t^{3\alpha}}{2^{2\alpha+1}[\Gamma(\alpha+1)]^2\Gamma(3\alpha+1)}\right\} \tag{19}$$

and so on. From eqs. (18) and (19) it is seen that the exact solution does not have a closed form. We also conclude that because of non-linearity of the problem. From numerical experiments view, the numerical values of approximate solutions for certain (x,t) combinations chosen inside the domain are calculated by using our Mathematica algorithm. Figure 1 shows first four approximate solution values and curves obtained by using with present FVIM. From those, it is also seen that each $u_n(x,t)$ solution is getting closer to the exact solution than $u_{n-1}(x,t)$.

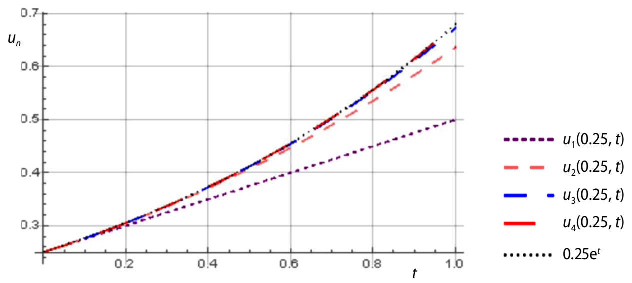

Figure 1. First four solution of FVIM for $\alpha=1$, $x=0.25$

Comparison of our data with those obtained other two methods (DTM and HPM) in [17, 18] can be seen from tab. 1.

While in [17, 18] authors found Taylor series expansion of exact solution, with optimized Lagrange multiplier FVIM gives one common iteration formula that generates successive approximate solutions without any known series format. In the fourth approximation,

Table 1. Comparison fourth solution of FVIM for $\alpha = 1$ with those obtained by HPM and DTM

x	t	Exact solution $u(x, t)$	$u_4(x, t)$	HPM and DTM $u_4(x, t)$	Abssolute error $u_4(x, t)$	Abssolute error HPM and DTM $u_4(x, t)$ [17, 18]
0.25	0.25	$3.2101\cdot10^{-1}$	$3.2101\cdot10^{-1}$	$3.21004\cdot10^{-1}$	$5.7849\cdot10^{-7}$	$2.1224\cdot10^{-6}$
0.25	0.5	$4.1218\cdot10^{-1}$	$4.1216\cdot10^{-1}$	$4.12109\cdot10^{-1}$	$2.00\cdot10^{-5}$	$7.09427\cdot10^{-5}$
0.25	0.75	$5.2925\cdot10^{-1}$	$5.2909\cdot10^{-1}$	$5.28687\cdot10^{-1}$	$1.6439\cdot10^{-4}$	$5.63481\cdot10^{-4}$
0.25	1.	$6.7957\cdot10^{-1}$	$6.7882\cdot10^{-1}$	$6.77083\cdot10^{-1}$	$7.5129\cdot10^{-4}$	$2.48712\cdot10^{-3}$
0.5	0.25	$6.4201\cdot10^{-1}$	$6.4201\cdot10^{-1}$	$6.42008\cdot10^{-1}$	$1.157\cdot10^{-6}$	$4.2448\cdot10^{-6}$
0.5	0.5	$8.2436\cdot10^{-1}$	$8.2432\cdot10^{-1}$	$8.24219\cdot10^{-1}$	$4.00\cdot10^{-5}$	$1.41885\cdot10^{-4}$
0.5	0.75	1.0585	1.0582	1.05737	$3.2879\cdot10^{-4}$	$1.12696\cdot10^{-3}$
0.5	1.	1.3591	1.3576	1.35417	$1.5026\cdot10^{-3}$	$4.97425\cdot10^{-3}$
0.75	0.25	$9.6302\cdot10^{-1}$	$9.6302\cdot10^{-1}$	$9.63013\cdot10^{-1}$	$1.7355\cdot10^{-6}$	$6.3672\cdot10^{-6}$
0.75	0.5	1.2365	1.2365	1.23633	$6.00\cdot10^{-5}$	$2.12828\cdot10^{-4}$
0.75	0.75	1.5878	1.5873	1.58606	$4.9318\cdot10^{-4}$	$1.69044\cdot10^{-3}$
0.75	1.	2.0387	2.0365	2.03125	$2.2539\cdot10^{-3}$	$7.46137\cdot10^{-3}$
1.	0.25	1.284	1.284	1.28402	$2.314\cdot10^{-6}$	$8.4896\cdot10^{-6}$
1.	0.5	1.6487	1.6486	1.64844	$8.00\cdot10^{-5}$	$2.83771\cdot10^{-4}$
1.	0.75	2.117	2.1163	2.11475	$6.5758\cdot10^{-4}$	$2.25392\cdot10^{-3}$
1.	1.	2.7183	2.7153	2.70833	$3.0052\cdot10^{-3}$	$9.9485\cdot10^{-3}$

while the maximum error obtained with FVIM, is read from tab. 1 as $3.00515594\cdot10^{-3}$, the maximum error obtained with HPM and DTM is read $9.94849513\cdot10^{-3}$ as three times bigger from that of FVIM. Furthermore, as it is expected, the minimum error occurs near the origin in a subregion of $[0,1]\times[0,1]$ and is approximately 10^{-7}.

In order to have closer numerical solutions to $u(x,t)$, residual method is going to be applied to $u_n(x,t)$ obtained from FVIM. According to section *Improvement of solutions obtained by FVIM*, error differential equation related with eq. (16):

$$
{}^{mRL}_{\ \ 0}\mathrm{D}_t^\alpha\left[e_n(x,t)\right]-\mathrm{D}_x^2 e_n(x,t)-\mathrm{D}_x e_n\left(x,\frac{t}{2}\right)e_n\left(\frac{x}{2},\frac{t}{2}\right)-\mathrm{D}_x e_n\left(x,\frac{t}{2}\right)u_4\left(\frac{x}{2},\frac{t}{2}\right)-
$$
$$
-\mathrm{D}_x u_4\left(x,\frac{t}{2}\right)e_n\left(\frac{x}{2},\frac{t}{2}\right)-\frac{1}{2}e_n(x,t)=-R_4(x,t) \tag{20}
$$

subject to $e_n(x,0)=0$ where $e_n(x,t)=u(x,t)-u_n(x,t)$ and $R_n(x,t)$ is residue function which:

$$
\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha}-\mathrm{D}_{xx}u_n(x,t)-\mathrm{D}_x u_n\left(x,\frac{t}{2}\right)u_n\left(\frac{x}{2},\frac{t}{2}\right)-\frac{1}{2}u_n(x,t)=R_n(x,t) \tag{21}
$$

With the same process in section *Process of fractional VIM (FVIM) for fractional PDE*, eq. (20) is solved functionally then with some values of (x,t) fig. 2 is plotted and tab. 2 is obtained.

– Case $\alpha = 0.9$, $\alpha = 0.8$, and $\alpha = 0.7$

For $\alpha = 0.9$, with modified Riemann-Liouville fractional derivative definition, the FVIM gives semi-analytical solutions which are calculated as functions of x and t. These solu-

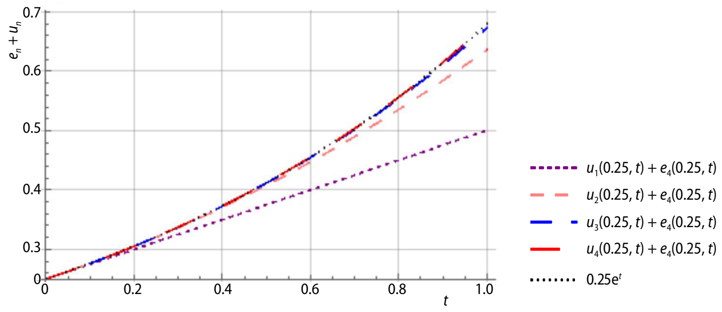

Figure 2. First four corrected solution obtained by FVIM for $\alpha = 1$, and exact solution of eq. (16)

Table 2. Comparison of $u_4(x,t)$ and $u_4(x,t) + e_4(x,t)$ improved approximate solutions of FVIM for $\alpha = 1$, and exact solution of eq. (16)

x	t	Exact solution $u(x, t)$	$u_4(x, t)$	$u_4(x, t) + e_4(x, t)$	Absolute error $u_4(x, t)$	Absolute error $u_4(x, t) + e_4(x, t)$
0.25	0.25	0.321006354	0.321005775	0.321006354	$5.78489726 \cdot 10^{-7}$	$5.11743958 \cdot 10^{-14}$
0.25	0.5	0.412180317	0.412160318	0.412180317	0.0000199999535	$2.76395965 \cdot 10^{-11}$
0.25	0.75	0.529250004	0.52908561	0.529250003	0.000164394395	$1.12232944 \cdot 10^{-9}$
0.25	1.0	0.679570457	0.678819168	0.679570441	0.000751288985	$1.58091924 \cdot 10^{-8}$
0.5	0.25	0.642012708	0.642011551	0.642012708	0.00000115697945	$1.02348791 \cdot 10^{-13}$
0.5	0.5	0.824360635	0.824320636	0.824360635	0.000039999907	$5.52791931 \cdot 10^{-11}$
0.5	0.75	1.0585	1.05817122	1.0585	0.000328788791	$2.24465888 \cdot 10^{-9}$
0.5	1.0	1.35914091	1.35763833	1.35914088	0.00150257797	$3.16183849 \cdot 10^{-8}$
0.75	0.25	0.963019063	0.963017327	0.963019063	0.00000173546918	$1.53523187 \cdot 10^{-13}$
0.75	0.5	1.23654095	1.23648095	1.23654095	0.0000599998606	$8.29187897 \cdot 10^{-11}$
0.75	0.75	1.58775001	1.58725683	1.58775001	0.000493183187	$3.36698833 \cdot 10^{-9}$
0.75	1.0	2.03871137	2.0364575	2.03871132	0.00225386695	$4.74275773 \cdot 10^{-8}$
1.0	0.25	1.28402541	1.2840231	1.28402541	0.0000023139589	$2.04697583 \cdot 10^{-13}$
1.0	0.5	1.64872127	1.64864127	1.64872127	0.0000799998141	$1.10558386 \cdot 10^{-10}$
1.0	0.75	2.11700001	2.11634244	2.11700001	0.000657577582	$4.48931777 \cdot 10^{-9}$
1.0	1.0	2.71828183	2.71527667	2.71828176	0.00300515594	$6.32367698 \cdot 10^{-8}$

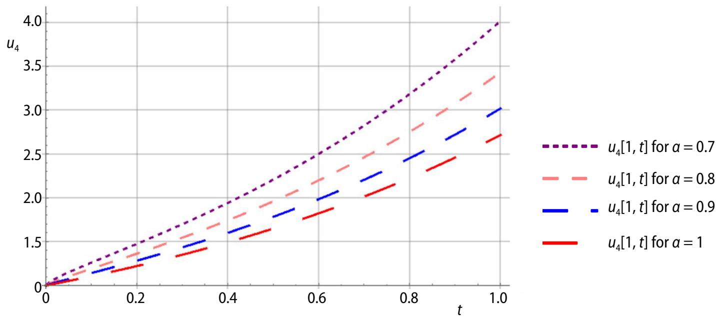

Figure 3. Forth order approximate solutions $u_4(x,t)$ obtained by using FVIM for α = 0.7, 0.8, 0.9, and 1 in eq. (16)

Table 3. Improved approximate solutions $u_4(x,t) + e_n(x,t)$ of eq. (16) obtained by FVIM for α = 0.7, 0.8, 0.9, and 1

x	t	$u_4(x, t) + e_2(x, t)$ for $\alpha = 0.7$	$u_4(x, t) + e_2(x, t)$ for $\alpha = 0.8$	$u_4(x, t) + e_2(x, t)$ for $\alpha = 0.9$	$u_4(x, t) + e_2(x, t)$ for $\alpha = 1$
0.25	0.25	$3.96187047 \cdot 10^{-1}$	$3.63028376 \cdot 10^{-1}$	$3.39067 \cdot 10^{-1}$	0.321006354
0.25	0.5	$5.52639137 \cdot 10^{-1}$	$4.90042012 \cdot 10^{-1}$	$4.45572 \cdot 10^{-1}$	0.412180317
0.25	0.75	$7.54664602 \cdot 10^{-1}$	$6.52387429 \cdot 10^{-1}$	$5.81521 \cdot 10^{-1}$	0.529250003
0.25	1.0	1.02089677	$8.63263195 \cdot 10^{-1}$	$7.56652 \cdot 10^{-1}$	0.679570441
0.5	0.25	$7.92374094 \cdot 10^{-1}$	$7.26056751 \cdot 10^{-1}$	$6.78134 \cdot 10^{-1}$	0.642012708
0.5	0.5	1.10527827	$9.80084024 \cdot 10^{-1}$	$8.91143 \cdot 10^{-1}$	0.824360635
0.5	0.75	1.5093292	1.30477486	1.16304	1.0585
0.5	1.0	2.04179353	1.72652639	1.5133	1.35914088
0.75	0.25	1.18856114	1.08908513	1.0172	0.963019063
0.75	0.5	1.65791741	1.47012604	1.33671	1.23654095
0.75	0.75	2.26399381	1.95716229	1.74456	1.58775001
0.75	1.0	3.0626903	2.58978959	2.26995	2.03871132
1.0	0.25	1.58474819	1.4521135	1.35627	1.28402541
1.0	0.5	2.21055655	1.96016805	1.78229	1.64872127
1.0	0.75	3.01865841	2.60954972	2.32608	2.11700001
1.0	1.0	4.08358707	3.45305278	3.02661	2.71828176

tions have several long terms, thus it is not written here. Instead, the obtained solutions for certain values are given in fig. 3 and tab. 3.

Example 2

The time fractional PDE is:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathrm{D}_{xx} u\left(x, \frac{t}{2}\right) u\left(x, \frac{t}{2}\right) - u(x,t) \qquad 0 < \alpha \le 1 \tag{22}$$

with initial condition $u(x,0) = x^2$, $(x,t) \in [0,1] \times [0,1]$ and the exact solution is $u(x,t) = x^2 \mathrm{e}^t$ when $\alpha = 1$, which was already solved by HPM in [18] and DTM in [17]. The numerical data are calculated separately in cases accordance with the value of α:

– Case $\alpha = 1$

So the equation becomes first order PDE with respect to t and the iteration formula for FVIM is constructed:

$$u_{n+1}(x,t) = u_n(x,t) -$$

$$-\frac{1}{\Gamma(\alpha+1)} \int_0^t \left[{}_0^{mRL}\mathrm{D}_\tau^\alpha u_n(x,\tau) - \mathrm{D}_{xx} u_n\left(x, \frac{\tau}{2}\right) u_n\left(x, \frac{\tau}{2}\right) - u_n(x,\tau) \right] (\mathrm{d}\tau)^\alpha \tag{23}$$

which will have a fixed point $u(x,t)$ taking initial approximate function $u_0(x,t) = x^2$. While n is increasing, approximate solutions of order n are indicated:

$$u_1(x,t) = x^2 \left[1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \right] \tag{24}$$

$$u_2(x,t)=x^2\left[1+\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{(2^{2-3\alpha}-2^{-2\alpha})\sqrt{\pi}t^{2\alpha}}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}+\frac{2\Gamma\left(\alpha+\frac{1}{2}\right)t^{3\alpha}}{\sqrt{\pi}2^{2\alpha+1}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right] \quad (25)$$

and so on. From eqs. (24) and (25) it is seen that the exact solution does not have a closed form. Following tab. 4 shows obtained approximate solutions and comparisons the values also obtained with other methods.

Table 4. Comparison fourth solution of FVIM for α = 1 with those obtained by HPM and DTM

x	t	Exact solution v(x,t)	$u_4(x, t)$	HPM and DTM $u_4(x, t)$	Absolute error $u_4(x, t)$	Absolute error HPM and DTM $u_4(x, t)$
0.25	0.25	0.08025158854298384	$8.02516\cdot10^{-2}$	$8.02511\cdot10^{-2}$	$1.385\cdot10^{-8}$	$5.306\cdot10^{-7}$
0.25	0.5	0.10304507941875801	$1.03044\cdot10^{-1}$	$1.03027\cdot10^{-1}$	$8.7944\cdot10^{-7}$	$1.77357\cdot10^{-5}$
0.25	0.75	0.13231250103829217	$1.32303\cdot10^{-1}$	$1.32172\cdot10^{-1}$	$9.8961\cdot10^{-6}$	$1.4087\cdot10^{-4}$
0.25	1.0	0.16989261427869032	$1.69838\cdot10^{-1}$	$1.69271\cdot10^{-1}$	$5.4648\cdot10^{-5}$	$6.21781\cdot10^{-4}$
0.5	0.25	0.32100635417193535	$3.21006\cdot10^{-1}$	$3.21004\cdot10^{-1}$	$5.5398\cdot10^{-8}$	$2.1224\cdot10^{-6}$
0.5	0.5	0.41218031767503205	$4.12177\cdot10^{-1}$	$4.12109\cdot10^{-1}$	$3.5178\cdot10^{-6}$	$7.09427\cdot10^{-5}$
0.5	0.75	0.5292500041531687	$5.2921\cdot10^{-1}$	$5.28687\cdot10^{-1}$	$3.9584\cdot10^{-5}$	$5.63481\cdot10^{-4}$
0.5	1.0	0.6795704571147613	$6.79352\cdot10^{-1}$	$6.77083\cdot10^{-1}$	$2.1859\cdot10^{-4}$	$2.48712\cdot10^{-3}$
0.75	0.25	0.7222642968868546	$7.22264\cdot10^{-1}$	$7.2226\cdot10^{-1}$	$1.2465\cdot10^{-7}$	$4.7754\cdot10^{-6}$
0.75	0.5	0.9274057147688222	$9.27398\cdot10^{-1}$	$9.27246\cdot10^{-1}$	$7.915\cdot10^{-6}$	$1.59621\cdot10^{-4}$
0.75	0.75	1.1908125093446296	1.19072	1.18954	$8.9065\cdot10^{-5}$	$1.26783\cdot10^{-3}$
0.75	1.0	1.5290335285082128	1.52854	1.52344	$4.9183\cdot10^{-4}$	$5.59603\cdot10^{-3}$
1.	0.25	1.2840254166877414	1.28403	1.28402	$2.2159\cdot10^{-7}$	$8.4896\cdot10^{-6}$
1.	0.5	1.6487212707001282	1.64871	1.64844	$1.4071\cdot10^{-5}$	$2.83771\cdot10^{-4}$
1.	0.75	2.117000016612675	2.11684	2.11475	$1.5834\cdot10^{-4}$	$2.25392\cdot10^{-3}$
1.	1.0	2.718281828459045	2.71741	2.70833	$8.7437\cdot10^{-4}$	$9.9485\cdot10^{-3}$

In order to improve semi analytic solutions $u_n(x,t)$, *i. e.*, to get it closer to exact solution we will apply residual method proposed in section *Improvement of solutions obtained by FVIM*. According to this section, error differential equation related with eq. (22):

$${}^{mRL}_{\ 0}\mathrm{D}_t^\alpha\left[e_n(x,t)\right]-\mathrm{D}_x^2 e_n\left(x,\frac{t}{2}\right)e_n\left(x,\frac{t}{2}\right)+u_4\left(x,\frac{t}{2}\right)+$$

$$+e_n(x,\mathrm{t})-e_n\left(\mathrm{x},\frac{t}{2}\right)\mathrm{D}_x^2 u_4\left(\mathrm{x},\frac{t}{2}\right)=-R_4(x,t) \quad (26)$$

subject to $e_n(x,0)=0$ where $R_n(\mathrm{x},\mathrm{t})$ is residue function and it is:

$$\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha}-\mathrm{D}_{xx}u_n(x,t)-\mathrm{D}_x u_n\left(x,\frac{t}{2}\right)u_n\left(\frac{x}{2},\frac{t}{2}\right)-\frac{1}{2}u_n(x,t)=R_n(x,t) \quad (27)$$

With the same process in section *Process of fractional VIM (FVIM) for fractional PDE*, eq. (26) is solved then with some values of (x,t) fig. 4 is plotted and tab. 5 is given.

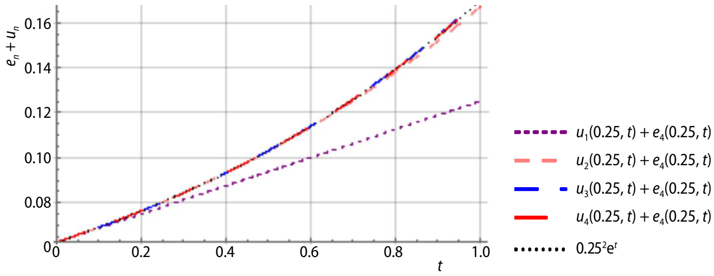

Figure 4. First four corrected solution obtained by FVIM for α = 1, and exact solution of eq. (22)

Table 5. Comparison of $u_4(x,t)$ and $u_4(x,t)+e_4(x,t)$ improved approximate solutions for α = 1, and exact solution of eq. (22)

x	t	x^2e^t	$u_4(x, t)$	$u_4(x, t) + e_4(x, t)$	Absolute error of $u_4(x, t)$	Absolute error of $u_4(x, t) + e_4(x, t)$
0.25	0.25	0.0802515885	0.0802515747	0.0802515885	$1.3849596\cdot10^{-8}$	$9.42849606\cdot10^{-15}$
0.25	0.5	0.103045079	0.1030442	0.103045079	$8.7944039\cdot10^{-7}$	$9.49285315\cdot10^{-12}$
0.25	0.75	0.132312501	0.132302605	0.1323125	0.00000989609653	$5.36436724\cdot10^{-10}$
0.25	1.0	0.169892614	0.169837966	0.169892605	0.0000546480454	$9.2993898\cdot10^{-9}$
0.5	0.25	0.321006354	0.321006299	0.321006354	$5.5398384\cdot10^{-8}$	$3.77139842\cdot10^{-14}$
0.5	0.5	0.412180317	0.4121768	0.412180317	0.00000351776156	$3.79714126\cdot10^{-11}$
0.5	0.75	0.529250004	0.52921042	0.529250002	0.0000395843861	$2.14574689\cdot10^{-9}$
0.5	1.0	0.679570457	0.679351865	0.67957042	0.000218592181	$3.71975592\cdot10^{-8}$
0.75	0.25	0.722264297	0.722264172	0.722264297	$1.24646364\cdot10^{-7}$	$8.48564646\cdot10^{-14}$
0.75	0.5	0.927405715	0.9273978	0.927405715	0.00000791496351	$8.54356784\cdot10^{-11}$
0.75	0.75	1.19081251	1.19072344	1.1908125	0.0000890648688	$4.82793051\cdot10^{-9}$
0.75	1.0	1.52903352	1.52854169	1.52903344	0.000491832408	$8.36945082\cdot10^{-8}$
1.0	0.25	1.28402541	1.28402519	1.28402541	$2.21593536\cdot10^{-7}$	$1.50855937\cdot10^{-13}$
1.0	0.5	1.64872127	1.6487072	1.64872127	0.0000140710462	$1.5188565\cdot10^{-10}$
1.0	0.75	2.11700001	2.11684168	2.117	0.000158337544	$8.58298758\cdot10^{-9}$
1.0	1.0	2.71828183	2.71740746	2.71828168	0.000874368726	$1.48790236\cdot10^{-7}$

Example 3

Finally, consider time fractional PDE:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathrm{D}_{xx} u\left(\frac{x}{2},\frac{t}{2}\right)\mathrm{D}_x u\left(\frac{x}{2},\frac{t}{2}\right) - \frac{1}{8}\mathrm{D}_x u(\mathrm{x},t) - u(x,t) \qquad 0<\alpha\le 1 \tag{28}$$

with initial condition $u(x,0)=x^2$, $(x,t)\in[0,1]\times[0,1]$ which was already solved by DTM in [17] and HPM in [18]. Its exact solution is $u(x,t)=x^2e^{-t}$ when $\alpha=1$. The numerical data are calculated separately at cases in accordance with the value of α as follows:

– Case $\alpha=1$

So the equation becomes first order PDE with respect to t and the iteration formula for FVIM is constructed:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha+1)}\int_0^t \Big[{}^{mRL}_{\ 0}\mathrm{D}^\alpha_\tau u_n(x,\tau) -$$

$$-D_{xx}u_n\left(\frac{x}{2},\frac{\tau}{2}\right)D_xu_n\left(\frac{x}{2},\frac{\tau}{2}\right)+\frac{1}{8}D_xu_n(x,\tau)+u_n(x,\tau)\Bigg](d\tau)^\alpha \tag{29}$$

which will have a fixed point $u(x,t)$ taking initial approximate function $u_0(x,t)=x^2$. While n is increasing, approximate solutions of order n are found:

$$u_1(x,t)=x^2\left[1-\frac{t^\alpha}{\Gamma(\alpha+1)}\right] \tag{30}$$

$$u_2(x,t)=x^2-\frac{x^2}{\Gamma(a+1)}t^a+\left(\frac{x^2}{\Gamma(2a+1)}-\frac{2^{-a-1}x}{\Gamma(2a+1)}+\frac{x}{4\Gamma(2a+1)}\right)t^{2a}+$$
$$+\frac{x\Gamma\left(a+\frac{1}{2}\right)}{12\sqrt{\pi}a^2\Gamma(a)\Gamma(3a)}t^{3a} \tag{31}$$

and so on. From eqs. (30) and (31) it is seen that the exact solution does not have a closed form, tab. 6.

Table 6. Comparison fourth solution of FVIM for $\alpha = 1$ with those obtained by HPM and DTM

x	t	Exact solution $u(x, t)$	Present method $u_4(x, t)$	HPM and DTM $u_4(x, t)$	Absolute error $u_4(x, t)$	Absolute error HPM and DTM $u_4(x, t)$
0.25	0.25	0.04867504	$4.8677\cdot10^{-2}$	$4.86755\cdot10^{-2}$	$1.9243\cdot10^{-6}$	$4.88167\cdot10^{-7}$
0.25	0.5	0.03790816	$3.79685\cdot10^{-2}$	$3.79232\cdot10^{-2}$	$6.0376\cdot10^{-5}$	$1.50109\cdot10^{-5}$
0.25	0.75	0.02952291	$2.99725\cdot10^{-2}$	$2.96326\cdot10^{-2}$	$4.4963\cdot10^{-4}$	$1.09659\cdot10^{-4}$
0.25	1.0	0.02299246	$2.4851\cdot10^{-2}$	$2.34375\cdot10^{-2}$	$1.8585\cdot10^{-3}$	$4.45035\cdot10^{-4}$
0.5	0.25	0.19470019	$1.94705\cdot10^{-1}$	$1.94702\cdot10^{-1}$	$4.4486\cdot10^{-6}$	$1.95267\cdot10^{-6}$
0.5	0.5	0.15163266	$1.51771\cdot10^{-1}$	$1.51693\cdot10^{-1}$	$1.3867\cdot10^{-4}$	$6.00434\cdot10^{-5}$
0.5	0.75	0.11809163	$1.19118\cdot10^{-1}$	$1.1853\cdot10^{-1}$	$1.0263\cdot10^{-3}$	$4.38635\cdot10^{-4}$
0.5	1.0	0.091969860	$9.61873\cdot10^{-2}$	$9.375\cdot10^{-2}$	$4.2175\cdot10^{-3}$	$1.78014\cdot10^{-3}$
0.75	0.25	0.43807544	$4.38083\cdot10^{-1}$	$4.3808\cdot10^{-1}$	$7.9494\cdot10^{-6}$	$4.39351\cdot10^{-6}$
0.75	0.5	0.34117349	$3.4142\cdot10^{-1}$	$3.41309\cdot10^{-1}$	$2.4698\cdot10^{-4}$	$1.35098\cdot10^{-4}$
0.75	0.75	0.26570618	$2.67529\cdot10^{-1}$	$2.66693\cdot10^{-1}$	$1.8223\cdot10^{-3}$	$9.86929\cdot10^{-4}$
0.75	1.0	0.20693218	$2.14399\cdot10^{-1}$	$2.10938\cdot10^{-1}$	$7.4665\cdot10^{-3}$	$4.00531\cdot10^{-3}$
1.	0.25	0.77880078	$7.78813\cdot10^{-1}$	$7.78809\cdot10^{-1}$	$1.2426\cdot10^{-5}$	$7.81068\cdot10^{-6}$
1.	0.5	0.60653065	$6.06916\cdot10^{-1}$	$6.06771\cdot10^{-1}$	$3.8532\cdot10^{-4}$	$2.40174\cdot10^{-4}$
1.	0.75	0.47236655	$4.75204\cdot10^{-1}$	$4.74121\cdot10^{-1}$	$2.8376\cdot10^{-3}$	$1.75454\cdot10^{-3}$
1.	1.0	0.36787944	$3.79485\cdot10^{-1}$	$3.75\cdot10^{-1}$	$1.1606\cdot10^{-2}$	$7.12056\cdot10^{-3}$

Now in order to approximate solutions $u_n(x,t)$ of FVIM get closer to exact solution we will apply residual method. According to section *Improvement of solutions obtained by FVIM*, error differential equation is:

$${}_0^{mRL}D_t^\alpha\left[e_n(x,t)\right]-D_x^2e_n\left(\frac{x}{2},\frac{t}{2}\right)D_xe_n\left(\frac{x}{2},\frac{t}{2}\right)-D_x^2e_n\left(\frac{x}{2},\frac{t}{2}\right)D_xu_2\left(\frac{x}{2},\frac{t}{2}\right)-$$

$$-\mathrm{D}_x e_n\left(\frac{x}{2},\frac{t}{2}\right)\mathrm{D}_x^2 u_2\left(\frac{x}{2},\frac{t}{2}\right)+\frac{1}{8}\mathrm{D}_x e_n(x,\mathrm{t})+e_n(x,\mathrm{t})=-R_n(x,t) \tag{32}$$

subject to $e_n(x,0)=0$ where $R_n(x,t)$ is residue function and it is:

$$\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha}-\mathrm{D}_{xx}u_n\left(\frac{x}{2},\frac{t}{2}\right)\mathrm{D}_x u_n\left(\frac{x}{2},\frac{t}{2}\right)+\frac{1}{8}\mathrm{D}_x u_n(x,t)+u_n(x,t)=R_n(x,t) \tag{33}$$

With the same process in section *Process of fractional VIM (FVIM) for fractional PDE*, eq. (32) is solved functionally, then fig. 5 is plotted with some values of (x,t) and the obtained values are given as in tab. 7.

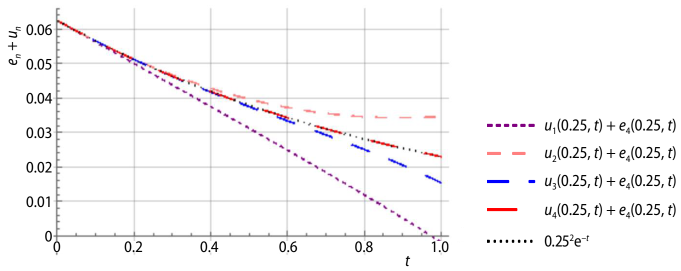

Figure 5. First four corrected solution obtained by FVIM for $\alpha = 1$, and exact solution of eq. (28)

Table 7. Comparison of $u_4(x,t)$ and $u_4(x,t)+e_4(x,t)$ improved approximate solutions for $\alpha = 1$, and exact solution of eq. (28)

x	t	Exact solution	$u_4(x, t)$	$u_4(x, t) + e_4(x, t)$	Absolute error $u_4(x, t)$	Absolute error $u_4(x, t) + e_4(x, t)$
0.25	0.25	0.0486750489	0.0486769732	0.0486750489	$1.92427039 \cdot 10^{-6}$	$9.56063336 \cdot 10^{-12}$
0.25	0.5	0.0379081662	0.0379685418	0.0379081711	$6.03756173 \cdot 10^{-5}$	$4.84764997 \cdot 10^{-9}$
0.25	0.75	0.0295229095	0.0299725366	0.0295230941	$4.49627111 \cdot 10^{-4}$	$1.84557960 \cdot 10^{-7}$
0.25	1.0	0.022992465	0.024850978	0.0229948993	$1.85851297 \cdot 10^{-3}$	$2.43426193 \cdot 10^{-6}$
0.5	0.25	0.194700195	0.194704644	0.194700195	$4.44864729 \cdot 10^{-6}$	$1.53773168 \cdot 10^{-11}$
0.5	0.5	0.151632665	0.151771333	0.151632672	$1.38668698 \cdot 10^{-4}$	$7.77544133 \cdot 10^{-9}$
0.5	0.75	0.118091638	0.119117951	0.118091933	$1.02631322 \cdot 10^{-3}$	$2.95237553 \cdot 10^{-7}$
0.5	1.0	0.0919698603	0.0961873186	0.0919737444	$4.21745828 \cdot 10^{-3}$	$3.88413411 \cdot 10^{-6}$
0.75	0.25	0.43807544	0.43808339	0.43807544	$7.94935900 \cdot 10^{-6}$	$2.24759161 \cdot 10^{-11}$
0.75	0.5	0.341173496	0.341420479	0.341173507	$2.46983482 \cdot 10^{-4}$	$1.13438472 \cdot 10^{-8}$
0.75	0.75	0.265706186	0.267528503	0.265706616	$1.82231695 \cdot 10^{-3}$	$4.29965939 \cdot 10^{-7}$
0.75	1.0	0.206932185	0.214398659	0.206937832	$7.46647344 \cdot 10^{-3}$	$5.64691542 \cdot 10^{-6}$
1.0	0.25	0.778800783	0.77881321	0.778800783	$1.24264055 \cdot 10^{-5}$	$3.08564312 \cdot 10^{-11}$
1.0	0.5	0.60653066	0.60691598	0.606530675	$3.85319968 \cdot 10^{-4}$	$1.55528676 \cdot 10^{-8}$
1.0	0.75	0.472366553	0.475204191	0.472367141	$2.83763830 \cdot 10^{-3}$	$5.88743118 \cdot 10^{-7}$
1.0	1.0	0.367879441	0.379484999	0.367887164	$1.16055584 \cdot 10^{-2}$	$7.72260585 \cdot 10^{-6}$

Conclusions

In this paper, time-fractional PDE with proportional delays are considered and their semi-analytical solutions are obtained by using FVIM composed of modified Riemann-Liouville type derivative. For $0<\alpha<1$, since the exact solutions of three test problems are not known, the residue error function is introduced additionally. With the aid of estimated error function it is also showed by figures and tables that the FVIM method yields sensitive values to the exact solutions or estimated errors of problems. Considering FVIM, the series solutions are found by using the initial conditions only. So the consecutive terms is transferring the data to next term and this is a significant advantage of the FVIM. If an exact solution exists for the equation, it can also be seen that the series solution converges to the closed form solution. On the other hand, it is observed from tables that are indicated for each case, that FVIM provides nearly exact solutions to problems with the same approximation order and same initial data.

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