NEW PERIODIC WAVE SOLUTIONS OF (3+1)-DIMENSIONAL SOLITON EQUATION

by

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In this paper, associating with the Hirota bilinear form, the three-wave method, which is applied to construct some periodic wave solutions of (3+1)-dimensional soliton equation, is a powerful approach to obtain periodic solutions for many non-linear evolution equations in the integrable systems theory.

Key words: *periodic wave solutions, three-wave method, Hirota bilinear form, (3+1)-dimensional soliton equation*

Introduction

It is famous that integrable systems exhibit rich variety of exact solutions, such as soliton, periodic, rational, and complexion solutions for PDE in mathematical physics [1-13]. Their exact solutions play an essential role in the proper understanding of qualitative features of the concerned phenomena and processes in different fields of non-linear science, such as non-linear optics, plasma physics and others. They could help us analyze the stability of these systems and validate the results of numerical analysis of non-linear PDE. Among them, the periodic solution is the one of the more important solutions to understand some of the natural phenomena. Here periodic solutions could be obtained through the Hirota bilinear form or their generalized counterparts in three-wave method [14]. The Hirota bilinear form plays an important role in mathematical physics and engineering fields. Once a non-linear PDE is written in bilinear form by a dependent variable's transformation, then multi-soliton solutions, rational solutions, Wronskian and Pfaffan forms of N-soliton solution could be obtained [15-18].

In this paper, we consider the (3+1)-dimensional soliton equation [17, 18]:

$$3u_{xz} - (2u_t + u_{xxx} - 2uu_x)_y + 2(u_x \partial_x^{-1} u_y)_x = 0 \tag{1}$$

Under the transformation:

$$u = -3(\ln f)_{xx} \tag{2}$$

we can change the (3+1)-dimensional soliton equation into the bilinear form:

$$(3\mathrm{D}_x\mathrm{D}_z - 2\mathrm{D}_y\mathrm{D}_t - \mathrm{D}_y\mathrm{D}_x^3) f f = 0 \tag{3}$$

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or, equivalently:

$$3ff_{xz}-3f_xf_z-2ff_{yt}+2f_yf_t+f_{xxx}f_y+3f_{xxy}f_x-3f_{xx}f_{xy}-ff_{xxxy}=0 \tag{4}$$

where $f=f(x,y,z,t)$ and the derivatives $\mathrm{D}_x,\mathrm{D}_y,\mathrm{D}_z$ are the Hirota operators [19] defined by:

$$D_x^aD_t^b(f\cdot g)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x^{'}}\right)^a\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t^{'}}\right)^b_{\times} f(x,t)g(x,t)|_{x=x^{'},t=t^{'}}$$

The (3+1)-dimensional soliton solution to eq. (1) was studied by many researchers in recent years. For example, its algebraic-geometrical solutions was explicitly given in the form of Riemann theta functions by using a non-linearized method of Lax pair. The N-soliton solution and its Wronskian form of solution have been discussed and derived by using the Hirota method and Wronskian technique [17]. The bilinear Backlund transformation and explicit solutions have also been obtained that is based on the Hirota bilinear method [20]. Some periodic wave solutions have been found in [21, 22].

The aim of this paper is to study the periodic wave solutions of (3+1)-dimensional soliton eq. (1), which is associated with the Hirota bilinear form. Then, with the aid of the corresponding graphic illustration, we would give a better understanding on the evolution of solutions of waves.

Describe the three-wave method

Here, we briefly show the three-wave method [14].

We now consider a non-linear PDE of the form:

$$p(u,u_x,u_t,u_y,u_z,u_{xx},...)=0 \tag{5}$$

where $u=u(x,y,z,t)$, p is a polynomial of u, and its various partial derivatives.

Step 1: By using the bell polynomial theories [23, 24], eq. (5) can be converted into Hirota bilinear form:

$$\mathrm{H}(\mathrm{D}_t,\mathrm{D}_x,\mathrm{D}_y,\mathrm{D}_z,...)ff=0 \tag{6}$$

where $f=f(x,y,z,t)$ and the derivatives $\mathrm{D}_x,\mathrm{D}_y,\mathrm{D}_z$ are the Hirota operators.

Step 2: Based on the Hitota bilinear of eq. (6), we assume that the solution can be expressed in the form:

$$f=a_1\cosh(tk_3+xk_1+yk_2+zk_4)+a_2\cos(tl_3+xl_1+yl_2+zk_4)+ \\ +a_3\cosh(tc_3+xc_1+yc_2+zc_4) \tag{7}$$

where $a_i(i=1,2,3)$, k_j,l_j, and $c_j(j=1,2,3,4)$ are all real parameters to be determined.

Step 3: Substituting eq. (7) into eq. (6) and collecting all the coefficients about:

$$\cosh(tk_3+xk_1+yk_2+zk_4),\ \sinh(tk_3+xk_1+yk_2+zk_4),\ \cos(tl_3+xl_1+yl_2+zl_4),$$
$$\sin(tl_3+xl_1+yl_2+zl_4),\ \cosh(tc_3+xc_1+yc_2+zc_4),\text{ and }\sinh(tc_3+xc_1+yc_2+zc_4)$$

we get the coefficients of these terms to zero and a set of algebraic equations contains parameters.

Step 4: Solving the set of algebraic equations in the *Step 3* using computer software, we obtain the values for involving parameters.

Step 5: Substituting these parameters into eq. (7) and corresponding transformation, we obtain the exact solution for original equation.

Next, we will show the three-wave method to solve the (3+1)-dimensional soliton for a PDE.

Periodic wave solutions of (3+1)-dimensional soliton solutions

Based on the known Hirota bilinear eq. (4) and transformation (2), we consider:

$$f = a_1 \cosh(tk_3 + xk_1 + yk_2 + zk_4) + a_2 \cos(tl_3 + xl_1 + yl_2 + zk_4) + \\ +a_3 \cosh(tc_3 + xc_1 + yc_2 + zc_4) \tag{8}$$

where $a_i (i = 1,2,3)$, k_j, l_j, and $c_j (j = 1,2,3,4)$ are all real parameters to be determined.

To get the periodic solutions, substituting (8) into eq. (4) and collecting all the coefficients about $\cosh(tk_3 + xk_1 + yk_2 + zk_4)$, $\sinh(tk_3 + xk_1 + yk_2 + zk_4)$, $\cos(tl_3 + xl_1 + yl_2 + zl_4)$, $\sin(tl_3 + xl_1 + yl_2 + zl_4)$, $\cosh(tc_3 + xc_1 + yc_2 + zc_4)$, $\sinh(tc_3 + xc_1 + yc_2 + zc_4)$, we can obtain a set of determining equations for a_i, k_j, l_j, and c_j:

$$-2a_1a_2(k_1^3l_2 + 3k_1^2k_2l_1 - 3k_1l_1^2l_2 - k_2l_1^3 - 3k_1l_4 + 2k_2l_3 + 2k_3l_2 - 3k_4l_1) = 0$$

$$-2a_1a_2(k_1^3k_2 - 3k_1^2l_2l_1 - 3k_1l_1^2k_2 + l_2l_1^3 - 3k_1k_4 + 2k_2k_3 \text{-} 2l_3l_2 + 3l_4l_1) = 0$$

$$2a_1a_3(c_1^3k_2 + 3c_1^2c_2k_1 + 3c_1k_1^2k_2 + c_2k_1^3 - 3c_1k_4 + 2c_2k_3 + 2c_3k_2 - 3c_4k_1) = 0$$

$$-2a_1a_3(c_1^3c_2 + 3c_1^2k_2k_1 + 3c_1k_1^2c_2 + k_2k_1^3 - 3k_1k_4 + 2k_2k_3 + 2c_3c_2 - 3c_4c_1) = 0$$

$$-2a_2a_3(c_1^3l_2 + 3c_1^2c_2l_1 - 3c_1l_1^2l_2 - c_2l_1^3 - 3c_1l_4 + 2c_2l_3 + 2c_3l_2 - 3c_4l_1) = 0$$

$$-2a_2a_3(c_1^3c_2 + 3c_1^2l_2l_1 - 3c_1l_1^2c_2 + l_2l_1^3 - 3c_1c_4 + 2c_2c_3 - 2l_3l_2 + 3l_4l_1) = 0$$

$$-2a_1^2(4k_1^3k_2 - 3k_1k_4 + 2k_2k_3) = 0, \quad -2a_2^2(4l_1^3l_2 + 3l_1l_4 - 2l_2l_3) = 0$$

$$2a_3^2(4c_1^3c_2 - 3c_1c_4 + 2c_2c_3) = 0$$

Solving this system of equations with the help of symbolic computation, we can present the following solutions of parameters:

– Case 1:

$$a_1 = 0, \quad a_2 = a_2, \quad a_3 = a_3, \quad c_1 = -\frac{l_1l_2}{c_2}, \quad c_2 = c_2, \quad c_3 = -\frac{l_1\left(3c_2^2l_2l_1^2 - l_1^2l_2^3 + 3c_2^2l_4\right)}{2c_2^3}$$

$$c_4 = \frac{c_2^2l_1^2l_2 + l_1^2l_2^3 + c_2^2l_4}{c_2l_2}, \quad k_1 = k_1, \quad k_2 = k_2, \quad k_3 = k_3, \quad k_4 = k_4, \quad l_1 = l_1, \quad l_2 = l_2$$

$$l_1 = l_1, \quad l_2 = l_2, \quad l_3 = l_3, \quad l_4 = l_4, \quad a_2 = 0, \quad k_1 = k_1$$

where $a_i (i = 1,3)$, c_2, $l_p (p = 1,2,3,4)$, and $k_j (j = 1,2,4)$ are arbitrary constants.

– Case 2:

$$a_1 = a_1, \quad a_2 = 0, \quad a_3 = a_3, \quad c_1 = \pm k_1, \quad c_2 = c_2, \quad c_3 = \mp\frac{k_1\left(4k_2k_1^2 - 3k_4\right)}{2k_2}, \quad c_4 = \frac{c_2k_4}{k_2}$$

$$k_1 = k_1, \quad k_2 = k_2, \quad k_3 = -\frac{k_1\left(4k_2k_1^2 - 3k_4\right)}{2k_2}, \quad k_4 = k_4, \quad l_1 = l_1, \quad l_2 = l_2, \quad l_3 = l_3$$

$$k_1 = k_1, \quad k_2 = k_2, \quad k_3 = -\frac{k_1\left(4k_2k_1^2 - 3k_4\right)}{2k_2}, \quad k_4 = k_4, \quad l_1 = l_1, \quad l_2 = l_2, \quad l_3 = l_3, \quad l_4 = l_4$$

where $a_i (i = 1,3)$, c_2, $l_p (p = 1,2,3,4)$, and $k_j (j = 1,2,4)$ are arbitrary constants.

– Case 3:

$$a_1 = a_1, \quad a_2 = 0, \quad a_3 = a_3, \quad c_1 = \pm k_1, \quad c_2 = c_2, \quad c_3 = \mp\frac{k_1\left(4c_2k_1^2 - 3c_4\right)}{2c_2}, \quad c_4 = c_4$$

$$k_1 = k_1, \quad k_2 = 0, \quad k_3 = -\frac{k_1\left(4c_2k_1^2 - 3c_4\right)}{2c_2}, \quad k_4 = 0, \quad l_1 = l_1, \quad l_2 = l_2, \quad l_3 = l_3, \quad l_4 = l_4$$

where $a_i (i = 1,3)$, c_2, $l_p (p = 1,2,3,4)$, and k_1 are arbitrary constants.

– Case 4:

$$a_1 = a_1, \quad a_2 = a_2, \quad a_3 = 0, \quad c_1 = c_1, \quad c_2 = c_2, \quad c_3 = c_3, \quad c_4 = c_4, \quad k_1 = -\frac{l_1l_2}{k_2}$$

$$k_2 = k_2, \quad k_3 = -\frac{l_1\left(3k_2^2l_1^2l_2 - l_1^2l_2^3 + 3k_2^2l_4\right)}{2k_2^3}, \quad k_4 = \frac{k_2^2l_1^2l_2 + l_1^2l_2^3 + k_2^2l_4}{k_2l_2}$$

$$l_1 = l_1, \quad l_2 = l_2, \quad l_3 = \frac{l_1\left(4l_1^2l_2 + 3l_4\right)}{2l_2}, \quad l_4 = l_4$$

where $a_i (i = 1,3)$, k_2, $c_j (j = 1,2,3,4)$, and $l_p (p = 1,2,4)$ are arbitrary constants.

– Case 5:

$$a_1 = a_1, \quad a_2 = a_2, \quad a_3 = a_3, \quad c_1 = -\frac{l_1l_2}{c_2}, \quad c_2 = c_2, \quad c_3 = -\frac{l_1\left(3c_2^2l_1^2l_2 - l_1^2l_2^3 + 3c_2^2l_4\right)}{2c_2^3}$$

$$c_4 = \frac{c_2^2l_1^2l_2 + l_1^2l_2^3 + c_2^2l_4}{c_2l_2}, \quad k_1 = -\frac{l_1l_2}{c_2}, \quad k_2 = c_2, \quad k_3 = -\frac{l_1\left(3c_2^2l_1^2l_2 - l_1^2l_2^3 + 3c_2^2l_4\right)}{2c_2^3}$$

$$k_4 = \frac{c_2^2l_1^2l_2 + l_1^2l_2^3 + c_2^2l_4}{c_2l_2}, \quad l_1 = l_1, \quad l_2 = l_2, \quad l_3 = \frac{l_1\left(4l_1^2l_2 + 3l_4\right)}{2l_2}, \quad l_4 = l_4$$

where $a_i (i = 1,3)$, c_2, and $l_p (p = 1,2,4)$ are arbitrary constants.

Thus, we can obtain the solutions of eq. (1):

$$u = -3\left(\ln f\right)_{xx} \tag{9}$$

where

$$f = a_1\cosh\left(tk_3 + xk_1 + yk_2 + zk_4\right) + a_2\cos\left(tl_3 + xl_1 + yl_2 + zl_4\right) + \\ + a_3\cosh\left(tc_3 + xc_1 + yc_2 + zc_4\right)$$

$a_i (i = 1,3)$, k_j, l_j, and $c_j (j = 1,2,3,4)$ are given in the Cases 1 to 6.

Now, we present graphic state of some special solutions. Here we mainly analyze the Cases 1 and 6.

– Case 1: At first, we give one choice of the parameters:

$$a_2 = 2, \quad a_3 = 3, \quad c_2 = 1, \quad l_1 = l_1, \quad l_2 = -1, \quad l_4 = 3$$

Then, we could obtain the following periodic solution:

$$u = \frac{6\cos\left(\dfrac{5}{2t} - x + y - 3z\right) - 9\cosh\left(\dfrac{7}{2t} - x - y + z\right)}{2\cos\left(\dfrac{5}{2t} - x + y - 3z\right) + 3\cosh\left(\dfrac{7}{2t} - x - y + z\right)} + \\ + \frac{3\left[\sin\left(\dfrac{5}{2t} - x + y - 3z\right) - 3\sinh\left(\dfrac{7}{2t} - x - y + z\right)\right]^2}{\left[2\cos\left(\dfrac{5}{2t} - x + y - 3z\right) + 3\cosh\left(\dfrac{7}{2t} - x - y + z\right)\right]^2} \tag{10}$$

Their plots when $y = 1$ and $t = 0, 3, 5$ are depicted in fig. 1, respectively, and the wave along different axis is shown in fig. 2.

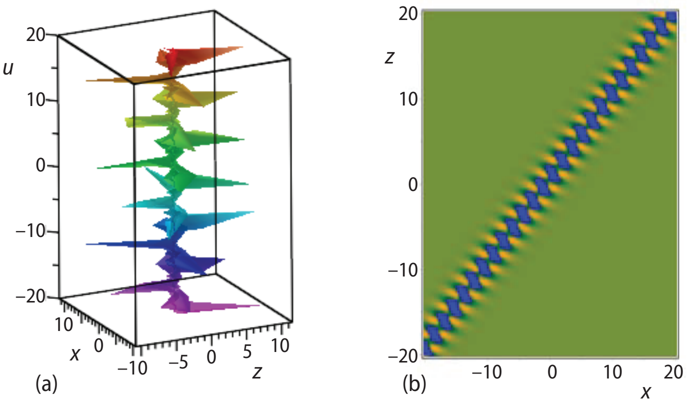

Figure 1. The periodic solution of eq. (10) with parameters $y = 1$, $t = 0$; (a) perspective view of the wave and (b) 2D-density plot
(*for color image see journal web site*)

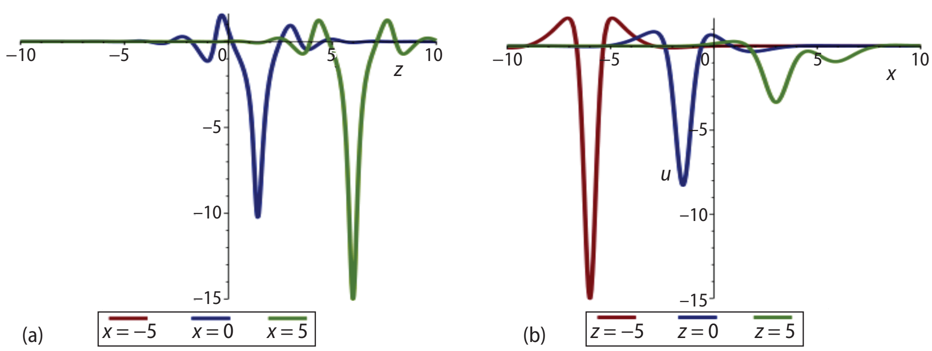

Figure 2. The periodic solution of eq. (10) with parameters $y = 1$, $t = 0$: The wave along the z-axis (a) and the wave along the x-axis (b)
(*for color image see journal web site*)

– Case 6: Second, we give one choice of the parameters:

$$a_1 = a_1,\ a_2 = 2,\ a_3 = 3,\ c_2 = 1,\ l_1 = 1,\ l_2 = -1,\ l_4 = 3$$

Then, we could obtain the following periodic solution:

$$u = -\frac{12\cosh\left(\frac{7}{2t} - x - y + z\right) - 6\cos\left(\frac{5}{2t} - x + y - 3z\right)}{4\cosh\left(\frac{7}{2t} - x - y + z\right) + 2\cos\left(\frac{5}{2t} - x + y - 3z\right)} + \\ + \frac{3\left[-4\sinh\left(\frac{7}{2t} - x - y + z\right) + 2\sin\left(\frac{5}{2t} - x + y - 3z\right)\right]^2}{\left[4\cosh\left(\frac{7}{2t} - x - y + z\right) + 2\cos\left(\frac{5}{2t} - x + y - 3z\right)\right]^2} \tag{11}$$

Their plots when $y = 1$ and $t = 0, 3, 5$ are depicted in figs. 3 and 4, respectively.

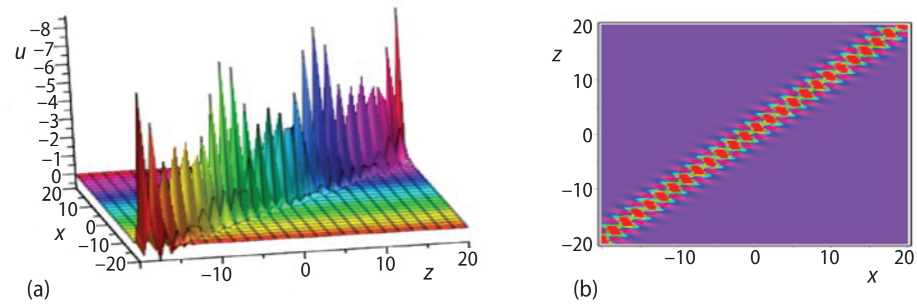

Figure 3. The periodic solution of eq. (11) with parameters $y = 1$, $t = 5$; (a) Perspective view of the wave and (b) 2D-Density plot
(*for color image see journal web site*)

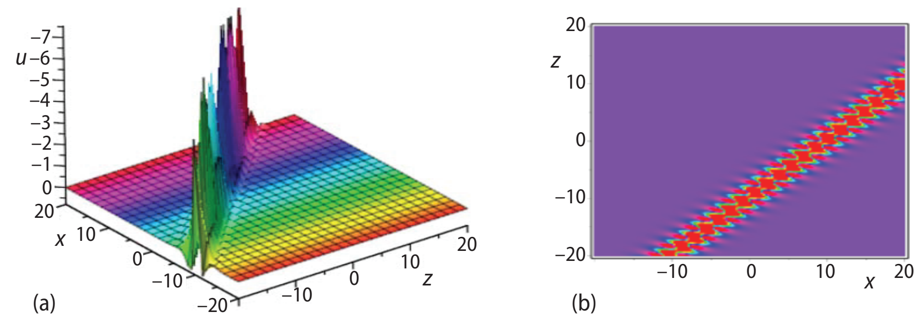

Figure 4. The periodic solution of eq. (11) with parameters $y = 1$, $t = 5$; (a) Perspective view of the wave and (b) 2D-Density plot
(*for color image see journal web site*)

Conclusion

In this paper, with the help of the Hirota bilinear form of the (3+1)-dimensional soliton equations, we presented the periodic wave solutions through the three-wave method. The new periodic wave solutions of the (3+1)-dimensional soliton equation are obtained and some special solutions were illustrated to help us be better to understand the evolution of solutions of waves. The performances of the methods afore mentioned are substantially influential and absolutely reliable for finding new exact solutions of other NPDE.

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Nomenclature

a_i (i = 1, 2, 3) – co-ordinates, [s^{-1}]
c_j (j = 1,2,3,4) – co-ordinates, [s^{-1}]
D_x, D_y, D_z – the Hirota operators, [K]
$f(x, y, z, t)$ – dependent variable, [K]
$g(x, y, z, t)$ – dependent variable, [K]
k_j (j = 1,2,3,4) – co-ordinates, [s^{-1}]
l_j (j = 1,2,3,4) – co-ordinates, [s^{-1}]
t – time co-ordinate, [s]
$u(x,y,z,t)$ – dependent variable, [K]
x, y, z – space co-ordinate, [m]

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