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INVERSE SCATTERING TRANSFORM FOR A NEW NON-ISOSPECTRAL INTEGRABLE NON-LINEAR AKNS MODEL

by

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Constructing integrable systems and solving non-linear partial differential equations are important and interesting in non-linear science. In this paper, Ablowitz-Kaup-Newell-Segur (AKNS)'s linear isospectral problem and its accompanied time evolution equation are first generalized by embedding a new non-isospectral parameter whose varying with time obeys an arbitrary smooth enough function of the spectral parameter. Based on the generalized AKNS linear problem and its evolution equation, a new non-isospectral Lax integrable non-linear AKNS model is then derived. Furthermore, exact solutions of the derived AKNS model is obtained by extending the inverse scattering transformation method with new time-varying spectral parameter. In the case of reflectinless potentials, explicit n-soliton solutions are finally formulated through the obtained exact solutions.

Key words: *non-isospectral integrable non-linear AKNS model, AKNS linear problem, Lax integrability, inverse scattering transformation method, soliton solution*

Introduction

There are two sets of non-linear PDE in soliton theory, they are the isospectral equations and non-isospectral equations. The isospectral PDE often describing solitary waves in lossless and uniform media, while the non-isospectral PDE describe the solitary waves in a certain type of non-uniform media. Starting from a proper linear spectral problem and its time evolution equation one can derive a whole hierarchy of isospectral PDE if the associated spectral parameter is independent of time. While non-isospectral non-linear PDE are usually resulted from the same linear spectral problem and its time evolution equation equipped with a time dependent spectral parameter.

The AKNS linear problem [1] consists of two parts, one is the spectral equation:

$$\phi_x = M\phi, \quad M = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{1}$$

the other is the time evolution equation of eigenfunction ϕ:

$$\phi_t = N\phi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2}$$

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where i is the imaginary unit, k is the spectral parameter independent of x, $q = q(x,t)$, $r = r(x,t)$ and their derivatives of any order with respect to x and t are smooth functions which vanish as x tends to infinity, and A, B, C are undetermined functions of x, t, q, r, and k. If a non-linear PDE is derived from the compatibility condition $\phi_{xt} = \phi_{tx}$ of eqs. (1) and (2), *i. e.*:

$$M_t - N_x + [M, N] = 0 \tag{3}$$

then the equation is called Lax integrable.

In 1991, Ablowitz and Clarkson [2] set $k_t = 0$ (isospectral) and then derived a hierarchy of isospectral integrable non-linear PDE from eq. (3), that is the famous AKNS isospectral hierarchy, which can be written [1]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -q \\ r \end{pmatrix}, \ (n = 0,1,2,\cdots) \tag{4}$$

Here the operator L is defined by:

$$L = \sigma \partial + 2 \begin{pmatrix} q \\ -r \end{pmatrix} \partial^{-1}(r, q), \quad \partial = \frac{\partial}{\partial x}, \quad \partial^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} \mathrm{d}x - \int_{x}^{+\infty} \mathrm{d}x \right), \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

It is easy to see that when $r = -1$ and $n = 2$ eq. (4) reduces to the celebrated Korteweg-de Vries (KdV) equation – a mathematical model of waves on shallow water surfaces:

$$q_t = q_{xxx} + 6qq_x \tag{6}$$

Letting $ik_t = (2ik)^n/2$ (non-isospectral), from eq. (3) one can derive a non-isospectral AKNS hiearchy [1]:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (n = 0,1,2,\cdots) \tag{7}$$

Recently, Zhang and Gao [3], Zhang and Li [4, 5] derived three non-isospectral AKNS systems:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty} \frac{1}{j!} L^j \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty} (-1)^j \frac{1}{(2j+1)!} L^{2j+1} \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \sum_{j=0}^{+\infty} (-1)^j L^j \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{10}$$

from eq. (3) equipped with three different non-isospectral parameters, k, which satisfy $ik_t = \mathrm{e}^{2ik}/2$, $ik_t = \sin(2ik)/2$, and $ik_t = (1 + 2ik)^{-1}/2$, respectively.

Since the non-isospectral PDE appearance, integrable equations have been substantially enriched. In the past several decades, constructing non-isospectral integrable non-linear PDE has attached much attention like those in [6-8]. In the present paper, on one hand we embed a non-isospectral parameter k satisfying:

$$ik_t = -f(ik) \tag{11}$$

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into eq. (3) for constructing a new and more general non-isospectral integrable non-linear AKNS model which includes infinite number of terms:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = -\sum_{j=0}^{+\infty} \frac{f^{(j)}(0)}{j!} L^j \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{12}$$

where $f(ik)$ is an arbitrary smooth enough function of ik. On the other hand, we exactly solve such a non-isospectral AKNS model (12) by extending the inverse scattering transformation (IST) method with new time-varying spectral parameter (11). It should be noted that eq. (12) is more general than eqs. (8)-(10). Especially, when $f(ik)$ is taken as $-e^{2ik}/2$, $-\sin(2ik)/2$ and $-(1+2ik)^{-1}/2$ eq. (11) gives the non-isospectral AKNS sytems (8)-(10), respectively.

As to what is integrable, there is no a uniform definination. The KdV eq. (6) is integrable in various senses such as Lax integrability, Liouville integrability and so on. There is a close relation between the existence of soliton solutions and the integrability of non-linear PDE, the known research results show that all the integrable systems exist soliton solutions [9]. From then when the soliton phenomena were first observed in 1834 and the KdV equation was exactly solved by the IST method [10], finding soliton solutions of non-linear PDE has become one of the most exciting and extremely active areas of research. With the development of fractional-order differential calculus [11-14], fractional PDE have been received more and more attention. In 2010, Fujioka *et al.* [15] described soliton propagation of an extended non-linear Schrodinger equation which incorporates fractional dispersion and a fractional non-linearity. Recently, Yang *et al.* [16] modeled fractal waves on shallow water surfaces via a local fractional KdV equation. Inspired by Gardner *et al.* [10] pioneer work in 1967, the IST method has developed to a systematic method for solving non-linear PDE and received a wide range of applications, see [1-8, 17, 18], for examples. In 1976, the IST was extended to non-isospectral integrable non-linear PDE [17]. Serkin *et al.* [18] found non-autonomous solitons in the framework of the IST with time-varying spectral parameter. However, to the best of our knowledge, the IST has not been extended to such a model (12) associated with the non-isospectral parameter (11).

Derivation and Lax integrability of new non-isospectral AKNS model

This section is based on the generalized AKNS linear spectral problem (1) and its time evolution eq. (2) equipped with a new non-isospectral parameter (11) to construct the new non-isospectral AKNS model (12). We have the following Theorem 1.

Theorem 1. The non-isospectral AKNS model (12) is Lax integrable, which can be derived from the compatibility condition (3) equipped with non-isospectral parameter (11) and the functions A, B, and C in eq. (2) are determined by:

$$A = \partial^{-1}(r, q)\begin{pmatrix} -B \\ C \end{pmatrix} + f(ik)x, \quad \begin{pmatrix} -B \\ C \end{pmatrix} = \sum_{s=1}^{+\infty} \begin{pmatrix} -b_s \\ c_s \end{pmatrix}(2ik)^{s-1} \tag{13}$$

$$\begin{pmatrix} -b_{s-1} \\ c_{s-1} \end{pmatrix} = L\begin{pmatrix} -b_s \\ c_s \end{pmatrix} - \frac{f^{(n-s+1)}(0)}{(n-s+1)!}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (s = 2,3,\cdots) \tag{14}$$

$$\begin{pmatrix} -b_1 \\ c_1 \end{pmatrix} = \frac{f^{(n)}(0)}{n!}\begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad n \to +\infty \tag{15}$$

Proof. Firstly, from eq. (3) we have:

$$A_x = qC - rB - ik_t, \quad q_t = B_x + 2ikB + 2qA, \quad r_t = C_x - 2ikC - 2rA \tag{16}$$

which can be rewritten:

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \lim_{n\to\infty} S_n, \quad S_n = \sum_{s=1}^{n} L \begin{pmatrix} -b_s \\ c_s \end{pmatrix}(2ik)^{n-s} - \sum_{s=1}^{n} \begin{pmatrix} -b_i \\ c_i \end{pmatrix}(2ik)^{n-s+1} - \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}(2ik)^j \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{17}$$

by using eqs. (13) and (14).

   We next substitute eq. (14) with the asymptotic condition (15) into eq. (17) and have:

$$\begin{pmatrix} -b_{s-1} \\ c_{s-1} \end{pmatrix} = L \begin{pmatrix} -b_s \\ c_s \end{pmatrix} - \frac{f^{(n-s+1)}(0)}{(n-s+1)!} \begin{pmatrix} -xq \\ xr \end{pmatrix}, \quad (s=2,3,\cdots), \quad S_n = -\sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} L^j \begin{pmatrix} -xq \\ xr \end{pmatrix} \tag{18}$$

by comparing the coefficients of $2ik$ in the second equation of eq. (17).

   Finally, from eq. (17) and the second equation of eq. (18) we obtain eq. (12). Thus, eq. (12) is a Lax integrable system and the proof is end.

Solutions through extending IST with new time-varying spectral parameter

   In this section, we solve the non-isospectral AKNS model (12) by extending the IST method with the new time varying spectral parameter (11). We have the following Theorems 2 and 3.

   Theorem 2. Supposing that the AKNS linear spectral problem (1) is equipped with the non-isospectral parameter (11), then its scattering data:

$$\left\{ \kappa_j(t), \quad c_j(t), \quad R(k,t) = \frac{a(k,t)}{b(k,t)}, \quad j=1,2,\cdots,n \right\} \tag{19}$$

$$\left\{ \overline{\kappa}_m(t), \quad \overline{c}_m(t), \quad \overline{R}(k,t) = \frac{\overline{a}(k,t)}{\overline{b}(k,t)}, \quad m=1,2,\cdots,\overline{n} \right\} \tag{20}$$

possess the following time dependence:

$$i\kappa_{jt}(t) = -f(i\kappa_j(t)), \quad c_j^2(t) = c_j^2(0) e^{\int_0^t f(2i\kappa_j(w))dw}, \quad a(k,t) = a(k,0), \quad b(k,t) = b(k,0) \tag{21}$$

$$i\overline{\kappa}_{mt}(t) = -f(i\overline{\kappa}_m(t)), \quad \overline{c}_m^2(t) = \overline{c}_m^2(0) e^{\int_0^t f(2i\overline{\kappa}_m(w))dw}, \quad \overline{a}(k,t) = \overline{a}(k,0), \quad \overline{b}(k,t) = \overline{b}(k,0) \tag{22}$$

where $c_j^2(0)$, $\overline{c}_m^2(0)$, $\kappa_j^2(0)$, $\overline{\kappa}_m^2(0)$, $R(k,0) = b(k,0)/a(k,0)$, $\overline{R}(k,0) = \overline{b}(k,0)/\overline{a}(k,0)$ are the scattering data of eq. (1) in the case of $(q(x,0), r(x,0))^T$.

   Theorem 3. Supposing that:

$$W(x,t) = E + P(x,t)P^T(x,t), \quad P(x,t) = \left[\frac{c_j(t)\overline{c}_m(t)}{\kappa_j - \overline{\kappa}_m} e^{i(\kappa_j - \overline{\kappa}_m)x} \right]_{\overline{n} \times n} \tag{23}$$

then the non-isospectral AKNS model (12) has the following *n*-soliton solutions:

$$q(x,t) = 2\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(x,t)), \quad r(x,t) = -\frac{\dfrac{d}{dx}\mathrm{tr}(W^{-1}(x,t)P(x,t)\dfrac{d}{dx}P^T(x,t))}{\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(x,t))} \tag{24}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a given matrix, E is a $\overline{n} \times \overline{n}$ unit matrix.

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Proof of Theorem 2. Since $P(x,k) = \phi_t(x,k) - N\phi(x,k)$ is also a solution of eq. (1), there exist two functions $\omega(t,k)$ and $\vartheta(t,k)$ so that:

$$\phi_t(x,k) - N\phi(x,k) = \gamma(k,t)\phi(x,k) + \tau(k,t)\tilde{\phi}(x,k) \tag{25}$$

where $\tilde{\phi}(x,k)$ also satisfies eq. (1) but is independent of $\phi(x,k)$.

Firstly, we take the discrete spectral $k = \kappa_j (\mathrm{Im}\,\kappa_j > 0)$. Since $\phi(x,\kappa_j)$ decays exponentially while $\tilde{\phi}(x,k)$ must increase exponentially as $x \to +\infty$, we then have $\tau(k,t) = 0$. Thus, eq. (25) is simplified:

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \gamma(\kappa_j,t)\phi(x,\kappa_j) \tag{26}$$

Using the inner product $(\phi_2(x,\kappa_j), \phi_1(x,\kappa_j))$ to left-multiply eq. (26) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_1(x,\kappa_j)\phi_2(x,\kappa_j) - [C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)] = 2\gamma(\kappa_j,t)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j) \tag{27}$$

Presuming $\phi(x,\kappa_j)$ to be the normalization eigenfunction in advance and noting that

$$2\int_{-\infty}^{\infty} c_j^2 \phi_1(x,\kappa_j)\phi_2(x,\kappa_j)\mathrm{d}x = 1$$

we have:

$$\gamma(\kappa_j,t) = -c_j^2 \int_{-\infty}^{\infty} [C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)]\mathrm{d}x$$

which can be rewritten as [3-5]:

$$\gamma(\kappa_j,t) = -c_j^2 ((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T, (B,C)^T) \tag{28}$$

From eq. (1) we have:

$$\phi_{1x}(x,\kappa_j) + i\kappa_j\phi_1(x,\kappa_j) = q(x)\phi_2(x,\kappa_j), \quad \phi_{2x}(x,\kappa_j) - i\kappa_j\phi_2(x,\kappa_j) = r(x)\phi_1(x,\kappa_j) \tag{29}$$

and hence obtain $[\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x = q(x)\phi_2^2(x,\kappa_j) + r(x)\phi_1^2(x,\kappa_j)$, the integration with respect to x from $-\infty$ to $+\infty$ of which gives:

$$\int_{-\infty}^{\infty} [q(x)\phi_2^2(x,\kappa_j) + r(x)\phi_1^2(x,\kappa_j)]\mathrm{d}x = \int_{-\infty}^{\infty} [\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)]_x \mathrm{d}x = 0 \tag{30}$$

In the other hand, we rewrite the second equation of eq. (14) as:

$$\begin{pmatrix} B \\ C \end{pmatrix} = -\lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}\frac{f^{(j)}(0)}{j!}\bar{L}^{j-s}\begin{pmatrix} xq \\ xr \end{pmatrix}(2i\kappa_j)^{s-1}, \quad \bar{L} = \sigma\partial - 2\begin{pmatrix} q \\ r \end{pmatrix}\partial^{-1}(-r,q) \tag{31}$$

and then obtain:

$$\gamma(\kappa_j,t) = \frac{1}{2}\lim_{n\to+\infty}\sum_{s=1}^{n}\sum_{j=s}^{n}\frac{f^{(j)}(0)}{j!}(2i\kappa_j)^{j-1} = \frac{1}{2}\lim_{n\to+\infty}\sum_{l=0}^{n-1}\frac{f^{(j)}(0)}{j!}(2i\kappa_j)^l = \frac{1}{2}f(2i\kappa_j) \tag{32}$$

through introducing the conjugation operator of $\bar{L}$:

$$\overline{L}^* = -\sigma \partial + 2 \begin{pmatrix} -r \\ q \end{pmatrix} \partial^{-1}(q,r), \quad \overline{L} = \sigma L \sigma \tag{33}$$

and using the result:

$$\left[\overline{L}^{*j-s}(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T, \begin{pmatrix} xq \\ xr \end{pmatrix} \right] = (2i\kappa_j)^{j-s} \left[(\phi_2^2(x,\kappa_j),\phi_1^2(x,\kappa_j))^T, \begin{pmatrix} xq \\ xr \end{pmatrix} \right] = -\frac{1}{2c_j^2(t)} \tag{34}$$

Thus, eq. (26) is simplified:

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \frac{1}{2}f(2i\kappa_j)\phi(x,\kappa_j) \tag{35}$$

Noting that:

$$N \rightarrow \begin{pmatrix} f(2i\kappa_j)x & 0 \\ 0 & -f(2i\kappa_j)x \end{pmatrix} \tag{36}$$

$$\phi(x,\kappa_j) \rightarrow c_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x}, \quad \phi_t(x,\kappa_j) \rightarrow c_{jt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x} + ix\kappa_{jt}c_j(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\kappa_j x} \tag{37}$$

as $x \rightarrow +\infty$, then from eqs. (35)-(37) we have:

$$i\kappa_{jt} = -f(2i\kappa_j), \quad c_{jt} = \frac{1}{2}f(2i\kappa_j)c_j \tag{38}$$

In a similar way, we obtain:

$$i\overline{\kappa}_{mt} = -f(2i\overline{\kappa}_m), \quad \overline{c}_{mt} = -\frac{1}{2}f(2i\overline{\kappa}_m)\overline{c}_m \tag{39}$$

Secondly, we consider k as a real continuous spectral. As did in [3-5], we can derive:

$$\frac{da(k,t)}{dt} = 0, \quad \frac{db(k,t)}{dt} = 0, \quad \frac{d\overline{a}(k,t)}{dt} = 0, \quad \frac{d\overline{b}(k,t)}{dt} = 0 \tag{40}$$

Finally, from eqs. (38)-(40) we arrive at eqs. (21) and (22). The proof is end.

Proof of Theorem 3. Given the scattering data (19)-(22) to the spectral problem (1), the non-isospectral AKNS model (12) has exact solutions:

$$q(x,t) = -2K_1(x,x,t), \quad r(x,t) = \frac{K_{2x}(x,x,t)}{K_1(x,x,t)} \tag{41}$$

where $K(x,y,t) = (K_1(x,y,t), K_2(x,y,t))^T$ satisfies Gel'fand-Levitan-Marchenko integral equation:

$$K(x,y,t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \overline{F}(x+y,t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \int_x^\infty F(z+x,t)\overline{F}(z+y,t)dz +$$

$$+ \int_x^\infty K(x,s,t) \int_x^\infty F(z+s,t)\overline{F}(z+y,t)dzds = 0 \tag{42}$$

with

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$$F(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\mathrm{e}^{ikx}\mathrm{d}k + \sum_{j=1}^{n} c_j^2 \mathrm{e}^{i\kappa_j x}, \quad \overline{F}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{R}(k,t)\mathrm{e}^{-ikx}\mathrm{d}k - \sum_{j=1}^{\overline{n}} \overline{c}_j^2 \mathrm{e}^{-i\overline{\kappa}_j x} \quad (43)$$

To construct soliton solutions, we set $R(k,t) = \overline{R}(k,t) = 0$. In this reflection less potentials case, as did in [3-5] we can obtain:

$$K_1(x,y,t) = -\mathrm{tr}(W^{-1}(x,t)\overline{\Lambda}(x,t)\overline{\Lambda}^T(y,t)), \quad K_2(x,y,t) = i\mathrm{tr}(W^{-1}(x,t)P(x,t)\Lambda(x,t)\overline{\Lambda}^T(y,t)) \quad (44)$$

where $\Lambda = (c_1(t)\mathrm{e}^{i\kappa_1 x}, c_2(t)\mathrm{e}^{i\kappa_2 x}, \cdots, c_n(t)\mathrm{e}^{i\kappa_n x})^T$, $\overline{\Lambda} = (\overline{c}_1(t)\mathrm{e}^{-i\overline{\kappa}_1 x}, \overline{c}_2(t)\mathrm{e}^{-i\overline{\kappa}_2 x}, \cdots, \overline{c}_n(t)\mathrm{e}^{-i\overline{\kappa}_n x})^T$.

Substituting eq. (44) into eq. (42), we finally obtain the *n*-soliton solutions (24). We therefore finish the proof.

Conclusion

In summary, we have generalized the AKNS spectral problem (1) and its evolution eq. (2) by embedding a non-isospectral parameter (11), which varies with time obeying an arbitrary smooth enough function of the spectral parameter. Starting from the generalized AKNS spectral problem (1) and its evolution eq. (2), together with (5), we constructed a new and more gerneral non-isospectral AKNS model (12) with infinite number of terms. In order to solve the derived non-isospectral AKNS model (12), the IST method with the new time-varying spectral parameter (11) is employed. As a result, exact solutions (41) are formulated and then reduced to explicit *n*-soliton solutions (24) in the case of reflectionless potentials. To the best of our knowledge, the derived non-isospectral AKNS model (12) and the obtained exact solutions (41) and *n*-soliton solutions (24) have not been reported in literatures.

Acknowledgment

Nomenclature

d/dt	–	first derivative, [–]
E	–	$\overline{n}\times\overline{n}$ unit matrix, [–]
e	–	base of natural logarithms, [–]
i	–	imaginary unit, [–]
j	–	natural number, [–]
k	–	spectral parameter, [–]
M, N	–	matrices, [–]

$n, \overline{n}$	–	positive integers, [–]
s	–	positive integer, [–]
T	–	transposition, [–]
t	–	time, [s]
x, y, z	–	displacements, [m]

Greek symbols

π	–	circumference ratio, [–]

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