A NEW TECHNIQUE FOR SOLVING THE 1-D BURGERS EQUATION

by

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In this paper, we address a new computational method, which is called the decomposition-Sumudu-like-integral-transform method, to handle the 1-D Burgers equation. The proposed method enables the efficient and accurate.

Key words: *analytic solution, Burgers equation, Adomian polynomials, decomposition-Sumudu-like-integral-transform method*

Introduction

The non-linear diffusion equation, structured by Burgers [1], was proposed for describing the turbulence [2], acoustic waves [3], thermo-viscous fluids [4], and water wave [5]. There are many computational methods for handling the problem, such as the variable separation method [6], explicit finite difference method [7], Cole-Hopf procedure transform [8], least-squares quadratic B-spline finite element method [9], shock-capturing finite difference method [10], tanh-coth method [11], Riccati equation rational expansion method [12], and others [13-20].

In this paper, we consider the following Burgers equation in 1-D case [21]:

$$\frac{\partial N(x,t)}{\partial t} + N(x,t)\frac{\partial^2 N(x,t)}{\partial x} = \lambda \frac{\partial^2 N(x,t)}{\partial x^2} \tag{1}$$

where λ is the diffusion constant, and $N(x, t)$ is the speed of the fluid flow in water wave.

The decomposition method was efficient and accurate for finding analytic solutions for the linear and non-linear PDE [22-25]. The Sumudu-like integral transform technique was proposed to handle the steady heat transfer problem [19]. The coupling technology for the decomposition and Sumudu-like integral transform methods have not yet proposed for finding the analytic solutions for the PDE.

Motivated by the aforementioned ideas, the main aim of the manuscript is to propose the decomposition-Sumudu-like-integral-transform method to solve the 1-D Burgers equation.

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The method proposed

In this section, we present the decomposition-Sumudu-like-integral-transform method used in the paper.

In order to illustrate the present technology, we consider the following PDE in the operator form:

$$DN(x,t)+RN(x,t)+\Pi N(x,t)=m(x,t) \tag{2}$$

with the initial condition:

$$N(x,0)=c(x) \tag{3}$$

where D is the first order linear differential operator, denoted by $D=\partial/\partial t$, R – the linear differential operator of less order than D, Π – the general non-linear differential operator, and $m(t)$ – the source term.

Adomian decomposition method

Following the idea [22-24], we find the inverse operator D^{-1} to both sides of (2) , and by using the initial condition, we have:

$$D^{-1}\left[DN(x,t)+RN(x,t)+\Pi N(x,t)\right]=D^{-1}\left[m(x,t)\right] \tag{4}$$

which reduces:

$$N(x,t)=\Phi(x,t)-D^{-1}\left[RN(x,t)\right]-D^{-1}\left[\Pi N(x,t)\right] \tag{5}$$

where

$$\Phi(x,t)=m(x,t)+c(x) \tag{6}$$

For the non-linear differential equations, the non-linear operator $\Lambda(N)=\Pi N$ can be represented by the Adomian polynomials, which are computed by [25]:

$$\Lambda\left[N(x,t)\right]=\Pi N(x,t)=\sum_{i=0}^{\infty}\Phi_i(x,t) \tag{7}$$

where

$$\Phi_i(x,t)=\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\lambda^i}\left\{\Pi\left[\lambda^i N_i(x,t)\right]\right\}_{\lambda=0}, \quad i=0,1,.... \tag{8}$$

Substituting eq. (7) into eq. (5), we obtain:

$$\sum_{i=0}^{\infty}N_i(x,t)=-D^{-1}\left[R\sum_{i=0}^{\infty}N_i(x,t)\right]-D^{-1}\left[\sum_{i=0}^{\infty}\Phi_i(x,t)\right] \tag{9}$$

where

$$N_0(x,t)=\Phi(x,t)=m(x,t)+c(x) \tag{10}$$

From eq. (9), we have the recursive relation:

$$N_i(x,t)=-D^{-1}\left[RN_i(x,t)\right]-D^{-1}\left[\Phi_i(x,t)\right] \tag{11}$$

where the initial condition is:

$$N_0(x,t)=\Phi(x,t)=m(x,t)+c(x) \tag{12}$$

When

$$\Lambda\left[N(x,t)\right] = \Pi N(x,t) = N(x,t)\frac{\partial N(x,t)}{\partial x} \tag{13}$$

we have the Adomian polynomials [25]:

$$\Phi_0(x,t) = N_0(x,t)\frac{\partial N_0(x,t)}{\partial x} \tag{14}$$

$$\Phi_1(x,t) = N_1(x,t)\frac{\partial N_0(x,t)}{\partial x} + N_0(x,t)\frac{\partial N_1(x,t)}{\partial x} \tag{15}$$

$$\Phi_2(x,t) = N_2(x,t)\frac{\partial N_0(x,t)}{\partial x} + N_1(x,t)\frac{\partial N_1(x,t)}{\partial x} + N_0(x,t)\frac{\partial N_2(x,t)}{\partial x} \tag{16}$$

$$\Phi_3(x,t) = N_3(x,t)\frac{\partial N_0(x,t)}{\partial x} + N_2(x,t)\frac{\partial N_1(x,t)}{\partial x} + N_1(x,t)\frac{\partial N_2(x,t)}{\partial x} + N_0(x,t)\frac{\partial N_3(x,t)}{\partial x} \tag{17}$$

With the aid of the Adomian polynomials, from eq. (11) we have the analytic solution of eq. (2):

$$N(x,t) = \sum_{i=0}^{\infty} N_i(x,t) \tag{18}$$

The Sumudu-like integral transform method

The Laplace-like integral transform of $\Lambda(t)$ is defined [19]:

$$\Lambda(\varpi) = \mathrm{Y}\left[\Lambda(t)\right] = \\ = \int_0^{\infty} \Lambda(t)\mathrm{e}^{-\frac{t}{\varpi}}\mathrm{d}t, \quad t > 0 \tag{19}$$

and its inverse Sumudu-like integral transform is defined [19]:

$$\Lambda(t) = \mathrm{Y}^{-1}\left[\Lambda(\varpi)\right] \tag{20}$$

The properties of the Sumudu-like integral transform [19] are listed in tab. 1.

Table 1. Properties of the Sumudu-like integral transform

Functions	The Laplace-like integral transforms
$\Lambda_1(t) \pm \Lambda_2(t)$	$\mathrm{Y}\left[\Lambda_1(t) \pm \Lambda_2(t)\right] = \Lambda_1(\varpi) \pm \Lambda_2(\varpi)$
$\Lambda(ct)$	$\mathrm{Y}\left[\Lambda(ct)\right] = \frac{1}{c}\Lambda\left(\frac{\varpi}{c}\right)$
$\Lambda^{(1)}(t)$	$\mathrm{Y}\left[\Lambda^{(1)}(t)\right] = \frac{1}{\varpi}\Lambda(\varpi) - \Lambda(0)$
$\int_0^t \Lambda(t)\mathrm{d}t$	$\mathrm{Y}\left[\int_0^t \Lambda(t)\mathrm{d}t\right] = \varpi\Lambda(\varpi)$
$\frac{t^i}{\Gamma(1+i)}$	$\mathrm{Y}\left[t^i\right] = \varpi^{1+i}$
e^{ct}	$\mathrm{Y}\left[\mathrm{e}^{ct}\right] = \frac{\varpi}{1 - c\varpi}$

The decomposition-Sumudu-like-integral-transform method

We derive the decomposition-Sumudu-like-integral-transform method based on the reported results.

Applying the Sumudu-like integral transform on two sides of eq. (2) with respect to t, we have:

$$\Upsilon\left[DN(x,t)+RN(x,t)+\Pi N(x,t)\right]=\Upsilon\left[m(x,t)\right] \tag{21}$$

such that:

$$\frac{1}{\varpi}N(x,\varpi)-N(x,0)+\Upsilon\left[RN(x,t)\right]+\Upsilon\left[\Pi N(x,t)\right]=m(x,\varpi) \tag{22}$$

which reduces to:

$$\frac{1}{\varpi}N(x,\varpi)-N(x,0)+RN(x,\varpi)+\Upsilon\left[\Pi N(x,t)\right]=m(x,\varpi) \tag{23}$$

We have:

$$N(x,\varpi)=\sum_{i=0}^{\infty}N_i(x,\varpi) \tag{24}$$

such that:

$$\sum_{i=0}^{\infty}N_i(x,\varpi)=-\varpi\left[R\sum_{i=0}^{\infty}N_i(x,\varpi)\right]-\varpi\left[\sum_{i=0}^{\infty}\Phi_i(x,\varpi)\right] \tag{25}$$

where

$$N_0(x,\varpi)=\Phi(x,\varpi)=m(x,\varpi)+c(x)\varpi \tag{26}$$

Thus, we have the recursive relation in the Sumudu-like-integral-transform form:

$$\Upsilon\left[N_i(x,t)\right]=-\Upsilon\left\{D^{-1}\left[RN_i(x,t)\right]\right\}-\Upsilon\left\{D^{-1}\left[\Phi_i(x,t)\right]\right\} \tag{27}$$

which leads to

$$N_i(x,\varpi)=-\varpi\left[RN_i(x,\varpi)\right]-\varpi\left[\Phi_i(x,\varpi)\right] \tag{28}$$

where the initial condition is

$$N_0(x,\varpi)=\Phi(x,\varpi)=m(x,\varpi)+c(x)\varpi \tag{29}$$

and

$$\Upsilon\left[\Phi_i(x,t)\right]=\Phi_i(x,\varpi) \tag{30}$$

When

$$\Upsilon\left\{\Lambda\left[N(x,t)\right]\right\}=\Upsilon\left[\Pi N(x,t)\right]=\Upsilon\left[N(x,t)\frac{\partial N(x,t)}{\partial x}\right] \tag{31}$$

we have the Adomian polynomials in the Sumudu-like-integral-transform form:

$$\Phi_0(x,\varpi)=\Upsilon\left[N_0(x,t)\frac{\partial N_0(x,t)}{\partial x}\right] \tag{32}$$

$$\Phi_1(x,\varpi)=\Upsilon\left[N_1(x,t)\frac{\partial N_0(x,t)}{\partial x}+N_0(x,t)\frac{\partial N_1(x,t)}{\partial x}\right] \tag{33}$$

$$\Phi_2(x,\varpi)=\Upsilon\left[N_2(x,t)\frac{\partial N_0(x,t)}{\partial x}+N_1(x,t)\frac{\partial N_1(x,t)}{\partial x}+N_0(x,t)\frac{\partial N_2(x,t)}{\partial x}\right] \tag{34}$$

$$\Phi_3(x,\varpi)=\Upsilon\left[\begin{array}{l}N_3(x,t)\dfrac{\partial N_0(x,t)}{\partial x}+N_2(x,t)\dfrac{\partial N_1(x,t)}{\partial x}+\\ +N_1(x,t)\dfrac{\partial N_2(x,t)}{\partial x}+N_0(x,t)\dfrac{\partial N_3(x,t)}{\partial x}\end{array}\right] \tag{35}$$

Thus, we have:

$$\Upsilon^{-1}\left[N_i(x,\varpi)\right]=-\Upsilon^{-1}\left\{\varpi\left[RN_i(x,\varpi)\right]\right\}-\Upsilon^{-1}\left\{\varpi\left[\Phi_i(x,\varpi)\right]\right\} \tag{36}$$

where the initial condition is:

$$\Upsilon^{-1}\left[N_0(x,\varpi)\right]=\Upsilon^{-1}\left[\Phi(x,\varpi)\right]=\Upsilon^{-1}\left[m(x,\varpi)+c(x)\varpi\right] \tag{37}$$

Finally, we obtain:

$$N(x,t)=\sum_{i=0}^{\infty}N_i(x,t)=\Upsilon^{-1}\left[\sum_{i=0}^{\infty}N_i(x,\varpi)\right] \tag{38}$$

Solving the 1-D Burgers equation

We now consider the 1-D Burgers eq. (1) with the initial condition:

$$c(x)=x^5 \tag{39}$$

From eqs. (1), (2), and (37), we have:

$$DN(x,t)=\frac{\partial N(x,t)}{\partial t}$$

$$RN(x,t)=-\lambda\frac{\partial^2 N(x,t)}{\partial x^2}$$

$$\Pi N(x,t)=N(x,t)\frac{\partial^2 N(x,t)}{\partial x}$$

$$m(x,t)=0$$

which yield the following recursive relation:

$$N_i(x,\varpi)=\varpi\lambda\frac{\partial^2 N_i(x,\varpi)}{\partial x^2}-\varpi\left[\Phi_i(x,\varpi)\right] \tag{40}$$

where

$$N_0(x,\varpi)=x^5\varpi \tag{41}$$

$$\Phi_0(x,\varpi)=\Upsilon\left[N_0(x,t)\frac{\partial N_0(x,t)}{\partial x}\right] \tag{42}$$

$$\Phi_1(x,\varpi)=\Upsilon\left[N_1(x,t)\frac{\partial N_0(x,t)}{\partial x}+N_0(x,t)\frac{\partial N_1(x,t)}{\partial x}\right] \tag{43}$$

$$\Phi_2\left(x,\varpi\right)=\mathrm{Y}\left[N_2\left(x,t\right)\frac{\partial N_0\left(x,t\right)}{\partial x}+N_1\left(x,t\right)\frac{\partial N_1\left(x,t\right)}{\partial x}+N_0\left(x,t\right)\frac{\partial N_2\left(x,t\right)}{\partial x}\right] \tag{44}$$

$$\Phi_3\left(x,\varpi\right)=\mathrm{Y}\left[\begin{array}{l}N_3\left(x,t\right)\dfrac{\partial N_0\left(x,t\right)}{\partial x}+N_2\left(x,t\right)\dfrac{\partial N_1\left(x,t\right)}{\partial x}+\\+N_1\left(x,t\right)\dfrac{\partial N_2\left(x,t\right)}{\partial x}+N_0\left(x,t\right)\dfrac{\partial N_3\left(x,t\right)}{\partial x}\end{array}\right] \tag{45}$$

Thus, we obtain the second component in the Sumudu-like-integral-transform form:

$$N_1\left(x,\varpi\right)=\varpi\lambda\frac{\partial^2 N_0\left(x,\varpi\right)}{\partial x^2}-\varpi\left[\Phi_0\left(x,\varpi\right)\right]=$$

$$=\varpi\lambda\frac{\partial^2\left(x^5\varpi\right)}{\partial x^2}-\varpi\mathrm{Y}\left[x^5\frac{\partial\left(x^5\right)}{\partial x}\right]=\varpi^2\left(20x^3-5x^9\right) \tag{46}$$

which leads to:

$$N_1\left(x,t\right)=t\left(20x^3-5x^9\right) \tag{47}$$

In a similar way, we give the third component in the Sumudu-like-integral-transform form:

$$N_2\left(x,\varpi\right)=\varpi\lambda\frac{\partial^2 N_1\left(x,\varpi\right)}{\partial x^2}-\varpi\left[\Phi_1\left(x,\varpi\right)\right]=$$

$$=\varpi\lambda\frac{\partial^2\left[\varpi^2\left(20x^3-5x^9\right)\right]}{\partial x^2}-\varpi\mathrm{Y}\left\{t\left(20x^3-5x^9\right)\frac{\partial\left[t\left(20x^3-5x^9\right)\right]}{\partial x}\right\}=$$

$$=\varpi^3\left(120x-360x^7\right)-\varpi^4\left(20x^3-5x^9\right)\left(60x^2-45x^8\right)=$$

$$=\varpi^3\left(120x-360x^7\right)-\varpi^4\left(120x^5-1200x^{11}-225x^{17}\right) \tag{48}$$

which leads to:

$$N_2\left(x,t\right)=\frac{t^2}{2}\left(120x-360x^7\right)-\frac{t^3}{6}\left(120x^5-1200x^{11}-225x^{17}\right) \tag{49}$$

Thus, we obtain the analytic solution of eq. (1):

$$N\left(x,t\right)=\sum_{i=0}^{\infty}N_i\left(x,t\right)=\mathrm{Y}^{-1}\left[\sum_{i=0}^{\infty}N_i\left(x,\varpi\right)\right]=$$

$$=x^5+t\left(20x^3-5x^9\right)+\frac{t^2}{2}\left(120x-360x^7\right)-\frac{t^3}{6}\left(120x^5-1200x^{11}-225x^{17}\right)+\ldots \tag{50}$$

Conclusion

In this work, we addressed the decomposition-Sumudu-like-integral-transform method, which is a coupling technique for the Adomian decomposition and Sumudu-like integral transform methods for the first time. The analytic solution for the 1-D Burgers equation was discussed in detail. The proposed method is efficient and accurate for finding the analytic solutions for the PDE in the water waves.

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Nomenclature

$N(x, t)$ – speed of the fluid flow, [ms^{-1}]
t – time co-ordinate, [s]
x – space co-ordinate, [m]

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