EXACT TRAVELING-WAVE SOLUTIONS FOR LINEAR AND NON-LINEAR HEAT TRANSFER EQUATIONS

by

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The exact traveling-wave solutions for the linear and non-linear heat transfer equations at several different excess temperatures are addressed and investigated in this paper.

Key words: *heat transfer equations, travelling-wave transformation, excess temperatures, exact solution*

Introduction

Ordinary differential equations (ODE) and partial differential equations (PDE) were used to describe the thermal problems in engineering sciences (see [1] and several related earlier references which are cited therein). Especially in heat transfer problems, the PDE [2] were adopted to govern the excess temperature fields in materials. In recent years, many different techniques were developed to derive the exact solutions for the heat transfer equations, such as the tanh method [3], exp-function method [4], (G'/G)-expansion method [5], heat-balance integral method [6], traveling-wave transformation method (TTM) [7, 8], and other methods [9-12].

However, the traveling-wave solutions of the heat transfer problems at several different excess temperatures have not yet been investigated. Motivated by the previous investigations, the aim of the present paper is to propose the traveling-wave solutions for the linear and nonlinear heat transfer equations.

The method applied

In order to introduce the concept of the traveling-wave solution, we consider the following PDE with respect to ξ and τ:

$$\aleph\left[\frac{\partial\Theta(\xi,\tau)}{\partial\tau},\frac{\partial^2\Theta(\xi,\tau)}{\partial\xi^2},\Theta^n(\xi,\tau)\right]=0 \tag{1}$$

where n is a positive integer.

Following the argument in [7, 8] , we set up the TTM, which is given by:

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$$\omega = \xi - \gamma\tau \tag{2}$$

where γ is a constant.

With the aid of the following chain rules:

$$\frac{\partial\Theta(\xi,\tau)}{\partial\tau} = -\gamma\frac{\partial\Theta(\omega)}{\partial\omega} \tag{3}$$

and

$$\frac{\partial^2\Theta(\xi,\tau)}{\partial\xi^2} = \frac{\partial^2\Theta(\omega)}{\partial\omega^2} \tag{4}$$

Equation (1) can be transformed into the ODE with respect to ω, which is given by:

$$\aleph\left[-\gamma\frac{\mathrm{d}\Theta(\omega)}{\mathrm{d}\omega}, \frac{\mathrm{d}^2\Theta(\omega)}{\mathrm{d}\omega^2}, \Theta^n(\omega)\right] = 0 \tag{5}$$

After obtaining the solution of eq. (5) by using the mathematical software, if we substitute eq. (2) into the obtained solution, we get the traveling-wave solution.

Traveling-wave solutions for linear and non-linear heat transfer problems

At first, we consider the linear heat transfer equation at the low excess temperature as follows, [2]:

$$\frac{\partial\Xi(\xi,\tau)}{\partial\tau} = \alpha\frac{\partial^2\Xi(\xi,\tau)}{\partial\xi^2} - \beta\,\Xi(\xi,\tau) \tag{6}$$

where α is the heat-diffusion coefficient, β – a constant, and $\Xi(\xi, \tau)$ – the excess temperature.

Following eq. (2), eq. (6) can be written:

$$-\gamma\frac{\mathrm{d}\Xi(\omega)}{\mathrm{d}\omega} = \alpha\frac{\mathrm{d}^2\Xi(\omega)}{\mathrm{d}\omega^2} - \beta\,\Xi(\omega) \tag{7}$$

With the help of the integrating-factor method [13] or the MATLAB software, the exact solution of eq. (7) is given by, [13]:

$$\Xi(\omega) = \begin{cases} \varpi_1 \mathrm{e}^{-k_1\omega} + \varpi_2 \mathrm{e}^{-k_2\omega}, \ \gamma^2 + 4\alpha\beta > 0, \\ (\varpi_3 + \varpi_4\omega)\mathrm{e}^{-\lambda\omega/2}, \ \gamma^2 + 4\alpha\beta = 0, \\ \mathrm{e}^{-\lambda\omega/2}(\varpi_5\cos\varphi\omega + \varpi_6\sin\varphi\omega), \ \gamma^2 + 4\alpha\beta < 0 \end{cases} \tag{8}$$

where ϖ_1, ϖ_2, ϖ_3, ϖ_4, ϖ_5 and ϖ_6 are constants, $\lambda = -\gamma / \alpha$, $k_1 = [-\gamma + (\gamma^2 + 4\alpha\beta)^{1/2}]/ 2\alpha$, $k_2 = [-\gamma - (\gamma^2 + 4\alpha\beta)^{1/2}] / 2\alpha$ and $\varphi = [-(\gamma^2 + 4\alpha\beta)]^{1/2} / 2\alpha$.

Substituting eq. (2) into eq. (8) , we obtain

$$\Xi(\xi,\tau) = \begin{cases} \varpi_1 \mathrm{e}^{-k_1(\xi-\gamma\tau)} + \varpi_2 \mathrm{e}^{-k_2(\xi-\gamma\tau)}, \ \gamma^2 + 4\alpha\beta > 0, \\ [\varpi_3 + \varpi_4(\xi-\gamma\tau)]\mathrm{e}^{-\lambda(\xi-\gamma\tau)/2}, \ \gamma^2 + 4\alpha\beta = 0, \\ \mathrm{e}^{-\lambda(\xi-\gamma\tau)/2}[\varpi_5\cos\varphi(\xi-\gamma\tau) + \varpi_6\sin\varphi(\xi-\gamma\tau)], \ \gamma^2 + 4\alpha\beta < 0 \end{cases} \tag{9}$$

The graphs of the traveling-wave solutions in eq. (6) are illustrated in figs. 1(a)-1(c).

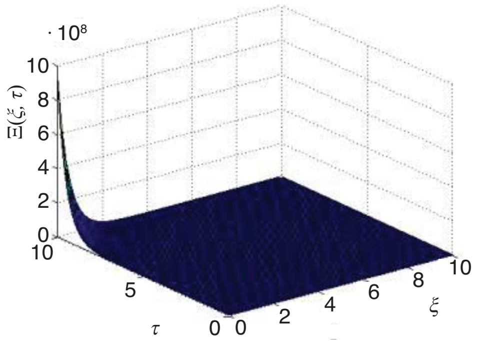

Figure 1(a). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta > 0$

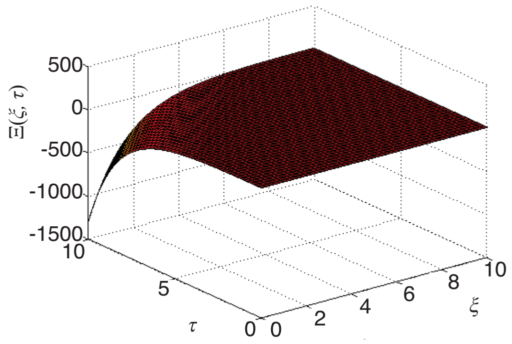

Figure 1(b). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta = 0$

As the second example, let us consider the following non-linear heat transfer equation at the high excess temperature (see [14, p. 159]):

$$\frac{\partial \Xi(\xi,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \Xi(\xi,\tau)}{\partial \xi^2} - \kappa \Xi^4(\xi,\tau) \tag{10}$$

where α is the thermal diffusivity, k – a constant, and $\Xi(\xi,\tau)$ – the excess temperature.

In view of eqs. (2)-(4), we can structure the non-linear ODE in the form:

$$-\gamma \frac{\mathrm{d}\Xi(\omega)}{\mathrm{d}\omega} = \alpha \frac{\mathrm{d}^2\Xi(\omega)}{\mathrm{d}\omega^2} - \kappa \Xi^4(\omega) \tag{11}$$

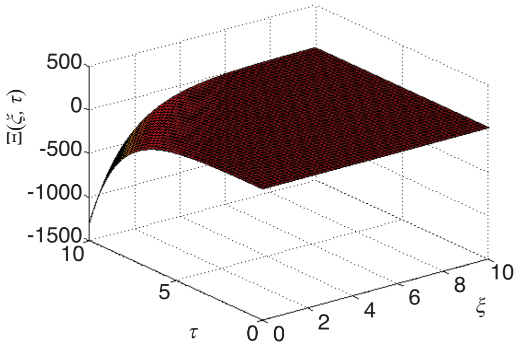

Figure 1(c). The traveling-wave solution for the linear heat transfer eq. (6) for $\gamma^2 + 4\alpha\beta < 0$

which leads to

$$\frac{\mathrm{d}^2\Xi(\omega)}{\mathrm{d}\omega^2} + \frac{\gamma}{\alpha}\frac{\mathrm{d}\Xi(\omega)}{\mathrm{d}\omega} - \frac{\kappa}{\alpha}\Xi^4(\omega) = 0 \tag{12}$$

With the aid of MATLAB software, the solution of eq. (12) can be written as:

$$\Xi(\omega) = \Lambda_1 \mathrm{e}^{-a\omega} - \frac{b\left(a^5\omega^5 - 5a^4\omega^4 + 20a^3\omega^3 - 60a^2\omega^2 + 120a\omega - 120\right) + \Lambda_2}{5a^6} \tag{13}$$

where Λ_1 and Λ_2 are two constants, $\alpha = \gamma/\alpha$ and $b = -k/\alpha$.

Thus, clearly, we easily obtain the traveling-wave solution for eq. (10) as:

$$\Xi(\xi,\tau) = \Lambda_1 \mathrm{e}^{-a(\xi-\gamma\tau)} - \frac{b}{5a^6}\left[a^5(\xi-\gamma\tau)^5 - 5a^4(\xi-\gamma\tau)^4 + 20a^3(\xi-\gamma\tau)^3\right] +$$
$$+ \frac{b}{5a^6}\left[60a^2(\xi-\gamma\tau)^2 - 120a(\xi-\gamma\tau) + 120\right] + \frac{\Lambda_2}{5a^6}. \tag{14}$$

The graphs of the traveling-wave solutions in eq. (10) are depicted in figs. 2(a)-2(c).

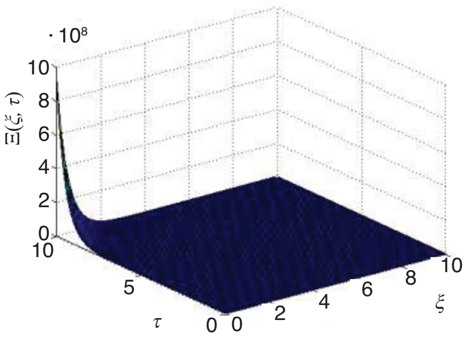

Figure 2(a). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 1$, $a = 1$, $b = -1$, $\gamma = 1$, and $A_2 = 0$

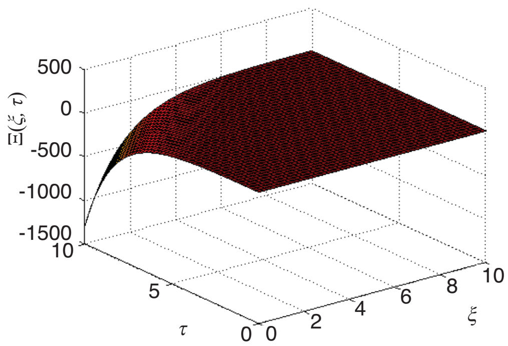

Figure 2(b). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 2$, $a = 1$, $b = -1$, $\gamma = 1$, and $A_2 = 0$

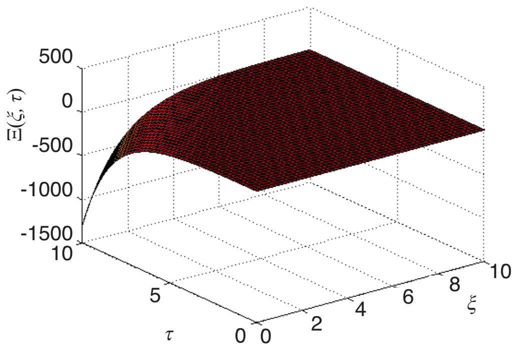

Figure 2(c). The traveling-wave solution for the non-linear heat transfer eq. (10) for the parameters $A_1 = 1$, $a = 1$, $b = -1$, $\gamma = 2$, and $A_2 = 0$

Conclusion

In our present work, we firstly investigated the linear and non-linear heat transfer equations at several different excess temperatures. With the help of the TTM, we transformed the linear and non-linear PDE arising in the heat transfer problems into the linear and non-linear ODE, respectively. We then obtained the solutions of the linear and nonlinear ODE by using the MATLAB software. Finally, the traveling-wave solutions of these heat transfer equations with the graphs are presented. The obtained results are given to reveal the efficiency of the techniques used in this paper.

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Nomenclature

α – heat-diffusion coefficient, [$Wm^{-1}K^{-1}$]
β – constant, [1/s]
κ – constant, [K^3s^{-1}]
ξ – space co-ordinate, [m]
$\Xi(\xi,\tau)$ – access temperature, [K]
τ – time co-ordinate, [s]

References

[1] Rohsenow, W. M., Choi, H. Y., *Heat, Mass, and Momentum Transfer*, Prentice-Hall, New York, USA, 1961

[2] Carslaw, H. S., *Introduction to the Mathematical Theory of the Conduction of Heat in Solids*, Macmillan, London, 2010

[3] Wazwaz, A. M., The Tanh Method for Generalized Forms of Nonlinear Heat Conduction and Burgers--Fisher Equations, *Applied Mathematics and Computation*, *169* (2005), 1, pp. 321-338

[4] Wen, Y.-X., Zhou, X.-W., Exact Solutions for the Generalized Nonlinear Heat Conduction Equations Using the Exp-Function Method, *Computers and Mathematics with Applications*, *58* (2009), 11-12, pp. 2464-2467

[5] Kabir, M. M., Analytic Solutions for Generalized Forms of the Nonlinear Heat Conduction Equation, *Nonlinear Analysis: Real World Applications*, *12* (2011), 5, pp. 2681-2691

[6] Hristov, J., Integral Solutions to Transient Nonlinear Heat (Mass) Diffusion with a Power-Law Diffusivity: a Semi-Infinite Medium with Fixed Boundary Conditions, *Heat and Mass Transfer*, *52* (2016), 3, pp. 635-655

[7] Schiesser, W. E., Griffiths, G. W., *A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab*, Cambridge University Press, Cambridge, UK, 2009

[8] Nikitin, A. G., Barannyk, T. A., Solitary Wave and other Solutions for Nonlinear Heat Equations, *Central European Journal of Mathematics,* 2 (2004), 5, pp. 840-858

[9] Yang, X.-J., A New Integral Transform Operator for Solving the Heat-Diffusion Problem, *Applied Mathematics Letters*, *64* (2017), Feb., pp. 193-197

[10] Yang, X.-J., A New Integral Transform Method for Solving Steady Heat-Ttransfer Problem, *Thermal Science*, *20* (2016), Suppl. 3, pp. S639-S642

[11] Yang, X.-J., A New Integral Transform with an Application in Heat-Transfer Problem, *Thermal Science*, *20* (2016), Suppl. 3, pp. S677-S681

[12] He, J.-H., Maximal Thermo-Geometric Parameter in a Nonlinear Heat Conduction Equation, *Bulletin of the Malaysian Mathematical Sciences Society, 39* (2016), 2, pp. 605-608

[13] Robinson, J. C., *An Introduction to Ordinary Differential Equations*, Cambridge University Press, Cambridge, UK, 2004

[14] Carslaw, H. S., Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, Oxford, UK, 1959

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