HEAT TRANSFER IN NATURAL CONVECTION FLOW OF NANOFLUID ALONG A VERTICAL WAVY PLATE WITH VARIABLE HEAT FLUX

by

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The present analysis is concerned to examine the enhancement of heat transfer in natural convection flow of nanofluid through a vertical wavy plate assumed at variable heat flux. The rate of heat transfer in nanofluid flow as compared to pure water can be increased due to increase the density of nanofluid which depends on the density and concentration of nanoparticles. For this analysis, Tiwari and Das model is used by considering two nanoparticles i. e. Al_2O_3 and Cu are suspended in a base fluid (water). A very efficient implicit finite difference technique converges quadratically is applied on the concerning PDE for numerical solution. The effects of pertinent parameters namely, volume fraction parameter of nanoparticle, wavy surface amplitude, Prandtl number and exponent of variable heat flux on streamlines, isothermal lines, local skin friction coefficient and local Nusselt number are shown through graphs. In this analysis, a maximum heat transfer rate is noted in Cu-water nanofluid through a vertical wavy surface as compared to Al_2O_3-water and pure water.

Key words: *natural convection flow, nanofluid, heat transfer, variable heat flux, numerical solution*

Introduction

Investigation of natural convection flow through a vertical sinusoidal wavy plate has significant importance in completely analyzing the industrial and engineering problems. Such flows are controlled by the density difference and, therefore, the flow has impact on heat transfer. A flow on the rough surface is disturbed by the surface and then flow and heat transfer rates are changed. These kind of roughened surfaces have many applications in real life which are taken into account such are flat plate condensers, heat transfer collectors, heat exchanger, and refrigerators. A well-known example of the mechanism of heat exchanger is a radiator which is commonly used in vehicle. In this mechanism, the generation of heat in engine can be shifted to air flowing through radiator. Yao [1] was the pioneer who studied heat transfer phenomena through vertical wavy plate in natural convection flow. He obtained numerical solution of the problem and found that heat transfer rate become double along wavy surface. Moulic and Yao [2] extended the work of Yao [1] by including free stream velocity in mixed convection flow through a

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wavy sheet. They concluded that heat flux through a flat plate is greater than that of wavy plate. Bhavnani and Bergles [3] performed an experimental study in natural convection flow through wavy plate and noted that heat transfer rate increases by about 15% when ratio between amplitude and wave length is 0.3. In another article Yao [4] examined free convection flow through a complex vertical wavy plate representing sinusoidal nature. He observed that heat transfer rate through a flat sheet was smaller as compared to complex wavy surface. Jang *et al.* [5] examined free convection flow through a vertical wavy plate and analysis both important phenomena of heat and mass transfer rates by calculating numerical solution. Jang and Yan [6] investigated transient aptitudes in natural convection flow for both heat and mass transfer through a complex vertical wavy plate. They applied a co-ordinate transformation for converting the wavy sheet to flat sheet and numerical solution was calculated. Few representative studies on natural convection flow through a wavy sheet can be seen in refs [7-11].

Flow along roughened surfaces with wall heat flux can be applied for cooling purpose of electronic devices and nuclear components [12]. Moulic and Yao [13] reported free convection flow through a rough vertical surface by considering constant heat flux. Tashtoush and Irshaid [14] considered a variable heat flux on a horizontal wavy surface. In this study, separation points were calculated by considering amplitude of a wavy surface as 0.2 and obtained solutions against x when exponent of variable heat flux is greater than –0.5. This work was extended by Tashtoush and Odat [15] by introducing magnetic field to the flow and found that the heat transfer rate is directly proportional to the increasing magnetic field strength. Rees and Pop [16] studied free convection flow through a vertical wavy sheet in porous medium by considering uniform heat flux. The work of Rees and Pop [16] extended by Molla *et al.* [17] introducing complex wavy surface and they found that additional harmonic substantially changes the temperature distribution and flow field near the surface. Rahman *et al.* [18] investigated free convection flow through a vertical wavy cone by considering temperature dependent viscosity and constant heat flux. Neagu [19] presented the analysis of the free convection flow through vertical wavy wall saturated in porous medium with heat and mass fluxes and obtained system was solved numerically.

A study of enhancement of heat transfer rate in fluids such as engine oil, ethylene glycol and water with low thermal conductivity has become an obstacle in engineering and industrial problems. To overcome these problems, many researchers worked to rise thermal conductivity of these kind of fluids. Experimental study reveals that thermal conductivity of these kind of fluids can be increased when nanoparticles are suspended with size up to 100 nm in these fluids and the resulting mixture is known as nanofluid. The heat transfer enhancement in nanofluids are very useful in many industrial problem such as pharmaceutical, food processing, fuel cells, microelectronics, hybrid-powered engines, and many others. Nanoparticles in nanomaterials are made by using metals Cu, Al, Au, Ag, and Fe, metal carbides SiC, oxides CuO, Al_2O_3, TiO_2, and nitrids SiN, AIN, *etc.* In the presence of these nanoparticles, thermal conductivity of nanofluid become greater as compared to base fluid. Choi [20] was the first who introduced the term of nanofluid. The analysis of heat transfer in nanofluids were investigated by few researchers [21-27]. A boundary-layer analysis in natural convection flow of a nanofluid through an isothermal vertical wavy surface was presented by Gorla and Kumari [28]. They considered Buongiorno's model and solved obtained equations numerically using implicit scheme. Mahdy and Ahmed [29] studied free convection laminar flow of nanofluid through a vertical wavy sheet saturated in porous medium. Srinivasacharya and Kumar [30] discussed mixed convection flow of a nanofluid on an inclined wavy sheet with radiation effect saturated in porous medium and solved numerically using chebyshev spectral collocation technique. They concluded that local heat transfer rate increases in both opposing and adding buoyancy flow cases with the increase of radiation. Habiba *et al.* [31] investigated a numerical study on free convection flow of Cu-water nanofluid through a vertical wavy sheet with uniform heat flux. Srinivasacharya and Kumar [32] investigated free convection flow of a nanofluid along an inclined wavy surface saturated in a porous medium in presence of wall heat flux and observed that mass and heat transfer rates in nanofluid increase by increasing the inclination of wavy surface.

The aim of current study is to investigate the rise of heat transfer rate in natural convection flow of nanofluids by considering Al_2O_3 and Cu nanoparticles through a vertical rough sheet with wall heat flux as a function of x. The solution of obtained equations is calculated numerically using Keller Box finite difference technique, and effects of emerging parameters on skin friction coefficient, streamlines, isotherm lines and local Nusselt number are presented graphically.

Mathematical formulation

The investigation on 2-D, steady natural convection flow of an incompressible nanofluid containing nanosize particles through a vertical wavy plate is carried out. The surface of wavy is considered as a sinusoidal function in term of sine is defined as $\overline{\sigma}(\overline{x}) = \overline{\alpha}\sin(2\pi\overline{x}/L)$ where $\overline{\alpha}$ and L are amplitude of the waviness and characteristic length of wavy plate respectively shown in fig. 1. A wavy surface is taken place along $\overline{x}$-axis and $\overline{y}$-axis is taken normal to it. The surface is considered to be heated due to variable flux $q_w = q_0(1+x^2)^m$ and applied behind the wavy surface, which was considered by Merkin and Mahmood [33], where q_o is constant and $x=\overline{x}/L$ is dimensionless variable.

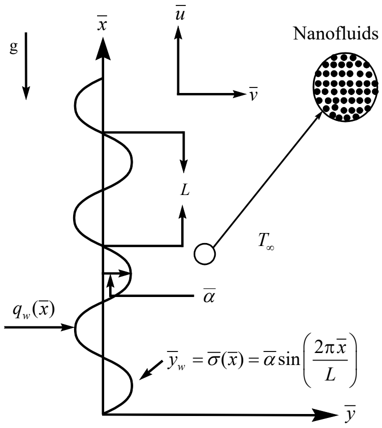

Figure 1. The physical model and co-ordinate system

Two kinds of nanoparticles namely, Al_2O_3 and Cu uniform size and spherical shape [21] are suspended in a base fluid. The material properti es of nanosize particles and base fluid are presented in tab. 1. In these properties, all physical properties except density of nanofluid do not depend on temperature. The governing PDE in nanofluid flow model presented by Tiwari and Das [22] in presence of Boussinesq approximations are defined:

Table 1. Material properties of nanoparticles and base fluid [24]

Properties	Fluid	Al_2O_3	Cu
c_p [Jkg^{-1}K^{-1}]	4179	765	385
ρ [kgm^{-3}]	997.1	3970	8933
k [Wm^{-1}K^{-1}]	0.613	40	400
β [K^{-1}]	$21\cdot10^{-5}$	$0.85\cdot10^{-5}$	$1.67\cdot10^{-5}$

$$\frac{\partial\overline{u}}{\partial\overline{x}} + \frac{\partial\overline{v}}{\partial\overline{y}} = 0 \tag{1}$$

$$\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}} + \overline{v}\frac{\partial\overline{u}}{\partial\overline{y}} = \frac{-1}{\rho_{\mathrm{nf}}}\frac{\partial\overline{p}}{\partial\overline{x}} + \frac{\mu_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}\nabla^2\overline{u} + \frac{\phi\rho_s\beta_s + (1-\phi)\rho_f\beta_f}{\rho_{\mathrm{nf}}}\mathrm{g}\left(T - T_\infty\right) \tag{2}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}=\frac{-1}{\rho_{\mathrm{nf}}}\frac{\partial \bar{p}}{\partial \bar{y}}+\frac{\mu_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}\nabla^2\bar{v} \tag{3}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}}+\bar{v}\frac{\partial T}{\partial \bar{y}}=\alpha_{\mathrm{nf}}\nabla^2 T \tag{4}$$

where the velocity components $\bar{u}$ and $\bar{v}$ are taken along $\bar{x}$- and $\bar{y}$-directions, respectively, μ_{nf} – the dynamic viscosity of nanofluid described by Brinkman [34], $\bar{p}$ – the pressure, ρ_f, β_f, and ρ_s, β_s – are densities and thermal expansion coefficients of water base fluid and nanoparticles, respectively, k_{nf} – the thermal conductivity and ρ_{nf} – the density, α_{nf} – the thermal diffusivity of nanofluid, g – the gravity, ϕ – the volume fraction parameter of nanoparticle, T – the temperature of nanofluid, T_∞ – the uniform ambient temperature, $(\rho c_p)_{\mathrm{nf}}$, $(\rho c_p)_f$, and $(\rho c_p)_s$ – are heat capacities of nanofluid, water base fluid and nanoparticle, respectively. The relations of k_{nf}, ρ_{nf}, μ_{nf}, $(\rho c_p)_{\mathrm{nf}}$, and α_{nf} are described [24]:

$$k_{\mathrm{nf}}=\frac{\left(k_s+2k_f\right)+2\phi\left(k_s-k_f\right)}{\left(k_s+2k_f\right)-\phi\left(k_s-k_f\right)}k_f,\quad \mu_{\mathrm{nf}}=\frac{\mu_f}{\left(1-\phi\right)^{2.5}}$$

$$\rho_{\mathrm{nf}}=\phi\rho_s+\left(1-\phi\right)\rho_f,\quad \alpha_{\mathrm{nf}}=\frac{k_{\mathrm{nf}}}{\left(\rho c_p\right)_{\mathrm{nf}}},\quad \left(\rho c_p\right)_{\mathrm{nf}}=\phi\left(\rho c_p\right)_s+\left(1-\phi\right)\left(\rho c_p\right)_f$$

In previously expression, k_f and μ_f are used to represent thermal conductivity and dynamic viscosity of water base fluid. The boundary conditions related to present flow geometry are given:

$$\begin{gathered}\bar{u}=\bar{v}=0,\quad \left(\bar{n}\bar{\nabla}T\right)=-\frac{q_w}{k_{\mathrm{nf}}}\quad \text{at}\quad \bar{y}=\bar{\sigma}\left(\bar{x}\right)\\ u\to 0,\quad T\to T_\infty,\quad \bar{p}\to p_\infty\quad \text{as}\quad \bar{y}\to\infty\end{gathered} \tag{5}$$

where p_∞ is pressure at ambient temperature and $\bar{n}$ is unit normal to surface. Upon introducing non-dimensional variables as considered by Pop *et al.* [35]:

$$u=\mathrm{Gr}^{-2/5}\frac{\rho_f L}{\mu_f}\bar{u},\quad v=\mathrm{Gr}^{-1/5}\frac{\rho_f L}{\mu_f}\left(\bar{v}-\sigma_x\bar{u}\right),\quad p=\mathrm{Gr}^{-4/5}\frac{L^2}{\rho_f {v_f}^2}\left(\bar{p}-p_\infty\right)$$

$$\theta=\left[\frac{T-T_\infty}{q_w(x)L}\right]k_f\,\mathrm{Gr}^{1/5},\quad x=\frac{\bar{x}}{L},\quad y=\mathrm{Gr}^{1/5}\frac{\bar{y}-\bar{\sigma}\left(\bar{x}\right)}{L},\quad \alpha=\frac{\bar{\alpha}}{L},\quad \sigma\left(x\right)=\frac{\bar{\sigma}\left(\bar{x}\right)}{L},\quad \frac{\mathrm{d}\sigma}{\mathrm{d}x}=\sigma_x$$

into eqs. (2)-(4) and neglecting the term of small order of Grashof number, the eqs. (2)-(4) in dimensionless form are written:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{h_2}\frac{\partial p}{\partial x}+\frac{1}{h_2}\sigma_x\,\mathrm{Gr}^{1/5}\frac{\partial p}{\partial y}+\frac{1}{h_1}\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}+\frac{h_3}{h_2}\theta \tag{6}$$

$$\sigma_x\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{1}{h_2}\mathrm{Gr}^{1/5}\frac{\partial p}{\partial y}+\frac{1}{h_1}\sigma_x\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}-\sigma_{xx}u^2 \tag{7}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+\frac{2mx}{\left(1+x^2\right)}u\theta=\frac{1}{h_4}\frac{k_{\mathrm{nf}}}{k_f}\frac{\left(1+\sigma_x^2\right)}{\mathrm{Pr}}\frac{\partial^2\theta}{\partial y^2} \tag{8}$$

The values of Grashof numer, h_1, h_2, h_3, and h_4 are given:

$$h_1=(1-\phi)^{2.5}\left[1-\phi+\phi\left(\frac{\rho_s}{\rho_f}\right)\right],\quad h_2=\left[1-\phi+\phi\left(\frac{\rho_s}{\rho_f}\right)\right],\quad h_3=\left[1-\phi+\phi\frac{(\rho\beta)_s}{(\rho\beta)_f}\right]$$

$$h_4=\left[1-\phi+\phi\frac{(\rho c_p)_s}{(\rho c_p)_f}\right],\quad \mathrm{Gr}=\frac{\mathrm{g}\beta_f q_w L^4}{k_f {\nu_f}^2}$$

The inviscid flow solution is used for calculating the pressure gradient along x-axis, which entails $\partial p/\partial x=0$. In eq. (7), the term on LHS is of $O(1)$, therefore, order of $\partial p/\partial y$ would be $O(\mathrm{Gr}^{-1/5})$. From eqs. (6) and (7), the term $\partial p/\partial y$ are eliminated, we get:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{h_1}\left(1+\sigma_x^2\right)\frac{\partial^2 u}{\partial y^2}-\frac{\sigma_x\sigma_{xx}}{1+\sigma_x^2}u^2+\frac{h_3}{h_2}\frac{\theta}{1+\sigma_x^2} \tag{9}$$

The boundary conditions in eq. (5) take the new form:

$$u=v=0,\quad \frac{\partial\theta}{\partial y}=-\frac{k_f}{k_{\mathrm{nf}}}\frac{1}{\sqrt{1+\sigma_x^2}}\quad \text{at}\quad y=0$$
$$u\to 0,\quad \theta\to 0,\quad p\to 0\quad \text{as}\quad y\to\infty \tag{10}$$

The following transformation are introduced for reducing the governing PDE (8)-(9) into a convenient form:

$$\eta=y(5x)^{-1/5},\quad \psi=(5x)^{4/5}f(x,\eta),\quad \theta(x,y)=(5x)^{1/5}\theta(x,\eta) \tag{11}$$

in which ψ represents stream function and expressed as $\partial\psi/\partial y=u$ and $\partial\psi/\partial x=-v$. The previous equations take form:

$$\frac{1}{h_1}\left(1+\sigma_x^2\right)f'''-\left(\frac{5x\sigma_x\sigma_{xx}}{1+\sigma_x^2}+3\right)f'^2+4ff''+\frac{h_3}{h_2}\frac{1}{\left(1+\sigma_x^2\right)}\theta=5x\left(f'\frac{\partial f'}{\partial x}-f''\frac{\partial f}{\partial x}\right) \tag{12}$$

$$\frac{1}{h_4}\frac{k_{\mathrm{nf}}}{k_f}\frac{\left(1+\sigma_x^2\right)}{\mathrm{Pr}}\theta''-\left(\frac{10mx^2}{1+x^2}+1\right)f'\theta+4f\theta'=5x\left(\frac{\partial\theta}{\partial x}f'-\frac{\partial f}{\partial x}\theta'\right) \tag{13}$$

and boundary conditions are:

$$f(x,\eta)=f'(x,\eta)=0,\quad \theta'(x,\eta)=-\frac{k_f}{k_{\mathrm{nf}}}\frac{1}{\sqrt{1+\sigma_x^2}}\quad \text{at}\quad \eta=0$$
$$f'(x,\eta)\to 0,\quad \theta(x,\eta)\to 0\quad \text{as}\quad \eta\to\infty \tag{14}$$

where prime signs denote partial derivative with respect to η. The skin friction coefficient C_{fx} and Nusselt number Nu_x are defined:

$$C_{fx} = \frac{\tau_w}{\rho_f U^2}, \quad \mathrm{Nu}_x = \frac{\bar{x} q_w(x)}{k_f (T_w - T_\infty)} \tag{15}$$

with wall shear stress τ_w and heat flux $q_w(x)$ are defined:

$$\tau_w = \mu_{\mathrm{nf}} \left(\bar{n}\, \bar{\nabla} \bar{u} \right)_{y=0}, \quad q_w = -k_{\mathrm{nf}} \left(\bar{n}\, \bar{\nabla} T \right)_{y=0} \tag{16}$$

The dimensionless form of skin friction coefficient and Nusselt number are obtained by substituting eqs. (11) and (16) into eq. (15), we get:

$$C_{fx}\,\mathrm{Gr}^{1/5} = \frac{(5x)^{2/5} \sqrt{1+\sigma_x^2}\, f''(x,0)}{(1-\phi)^{2.5}}, \quad \mathrm{Nu}_x (\mathrm{Gr}^{-1/5}) = \frac{x^{4/5}}{5^{1/5} \theta(x,0)}$$

Results and discussion

The solution of obtained system of non-linear PDE (12) and (13) subject to the boundary conditions (14) are calculated numerically using Keller Box implicit finite difference technique. For detailed study of the method, reader is referred to Cebeci and Bradshaw [36]. The computed results by present scheme for current flow problem are shown in tab. 2 and compared with that of previous results of Pop *et al.* [35] and Siddiqa *et al.* [10] for various values of *x* and found in good agreement, which shows the effectivity and accuracy of the method.

Table 2. Numerical results of $\theta(x, 0)$ for different values of x with $\phi = m = 0.0$ and Pr = 1.0

	$\alpha = 0.0$			$\alpha = 0.2$		
x	Present	[35]	[10]	Present	[35]	[10]
1.50	1.3574	1.3585	1.3574	1.5258	1.5217	1.5309
1.625	1.3574	1.3585	1.3574	1.5334	1.5307	1.5365
1.75	1.3574	1.3585	1.3574	1.5086	1.5137	1.5133
1.875	1.3574	1.3585	1.3574	1.5032	1.5009	1.5103
2.0	1.3574	1.3585	1.3574	1.5242	1.5201	1.5305
2.125	1.3574	1.3585	1.3574	1.5307	1.5279	1.5351
2.250	1.3574	1.3585	1.3574	1.5100	1.5147	1.5158
2.375	1.3574	1.3585	1.3574	1.5053	1.5029	1.5135
2.5	1.3574	1.3585	1.3574	1.5232	1.5191	1.5306
2.625	1.3574	1.3585	1.3574	1.5288	1.5261	1.5346
2.750	1.3574	1.3585	1.3574	1.5109	1.5154	1.5178
2.875	1.3574	1.3585	1.3574	1.5068	1.5043	1.5160
3.0	1.3574	1.3585	1.3574	1.5224	1.5184	1.5310

In this study Al_2O_3-water and Cu-water nanofluids are considered. Effects of pertinent parameters namely volume fraction parameter of nanoparticle, ϕ, Prandtl number, amplitude of wavy surface, α, and exponent of variable heat flux, m, on $C_{fx}\mathrm{Gr}^{1/5}$ (local skin friction coefficient) and $\mathrm{Nu}_x\mathrm{Gr}^{-1/5}$ (local Nusselt number) are presented through graphs. In figures, the solid and dashed lines are used to represent the behavior of Cu-water and Al_2O_3-water nanofluids, respectively. Also, the streamlines and isothermal lines are drawn which describes the complete mechanism of the present flow problem. Figures 2 and 3 illustrate the variations in $C_{fx}\mathrm{Gr}^{1/5}$ and $\mathrm{Nu}_x\mathrm{Gr}^{-1/5}$ against volume fraction parameter ϕ for both Cu-water and Al_2O_3-water nanofluids along vertical flat surface, $\alpha = 0$, and wavy surface, $\alpha = 0.1$. It is observed that both $C_{fx}\mathrm{Gr}^{1/5}$ and

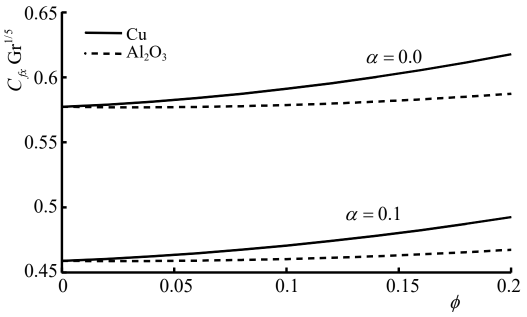

Figure 2. The variation of C_{fx}Gr$^{1/5}$ (local skin friction coefficient) against volume fraction parameter, ϕ, when $m = x = 1.0$ and Pr = 6.2 are fixed

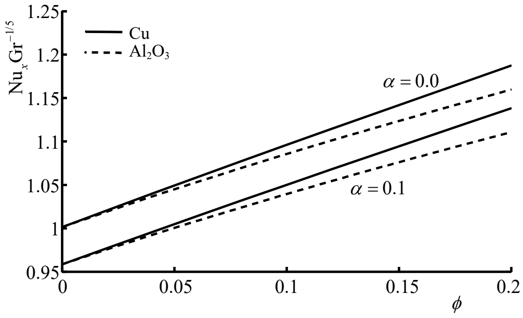

Figure 3. The variation of Nu$_x$ Gr$^{-1/5}$ (local Nusselt number) against volume fraction parameter, ϕ, when $m = x = 1.0$ and Pr = 6.2 are fixed

Nu$_x$Gr$^{-1/5}$ increase in both nanofluids with increasing the value of ϕ, which is due to the reason that the viscosity of nanofluid increases due to the enhancement of nanoparticle volume fraction ϕ. It is clear that the values of both C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ are higher in Cu-water nanofluid than the values of Al_2O_3-water nanofluid due to the highest viscosity and thermal conductivity of Cu-water nanofluid. It is also observed that flow and heat transfer rates in both nanofluids are larger for vertical flat surface ($\alpha = 0$) as compared to that of rough wavy surface ($\alpha = 0.1$).

Figsures 4 and 5 show the variations of C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ for both nanofluids against x along a wavy surface for some values of amplitude parameter, α. It is seen that C_{fx}-Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ increase along a wavy surface in upstream direction also both decrease with increase in the amplitude of wavy surface. This is because higher roughness of surface

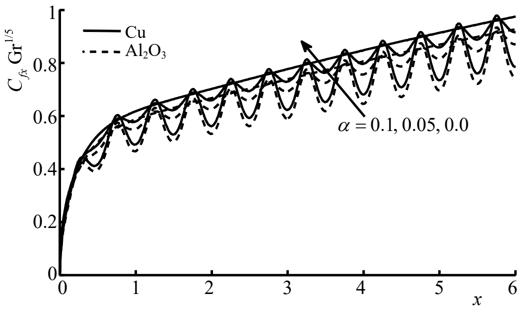

Figure 4. The variation of C_{fx}Gr$^{1/5}$ (local skin friction coefficient) against x, when $m = 1.0$, Pr = 6.2 and $\phi = 0.2$ are fixed

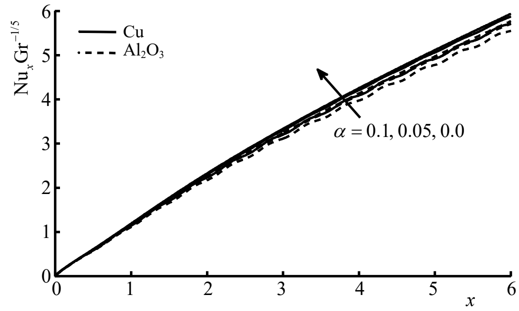

Figure 5: The variation of Nu$_x$Gr$^{-1/5}$ (local Nusselt number) against x, when $m = 1.0$, Pr = 6.2 and $\phi = 0.2$ are fixed

enhances the viscous effects within the boundary-layer and consequently flow and heat transfer rate are reduced. Also flow and heat transfer rates in Cu-water nanofluid is noted greater than that of Al_2O_3-water. Effects of exponent of variable heat flux, m, on C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ are presented in figs. 6 and 7, respectively. It is seen that C_{fx}Gr$^{1/5}$ decreases and Nu$_x$Gr$^{-1/5}$ increases against x with the increasing values of m along a wavy surface. This is because decrease of fluid temperature results in increase of heat transfer rate at wavy surface. It is also observed that the amplitude of the waves decreases the magnitude of C_{fx}Gr$^{1/5}$ and increases the magnitude of Nu$_x$Gr$^{-1/5}$. The values of Cu-water nanofluid in both C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ is greater than the

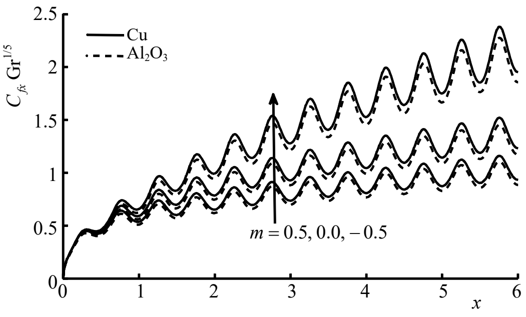

Figure 6. The variation of C_{fx}Gr$^{1/5}$ (local skin friction coefficient) against x, when ϕ = 0.2, Pr = 6.2 and α = 0.1 are fixed

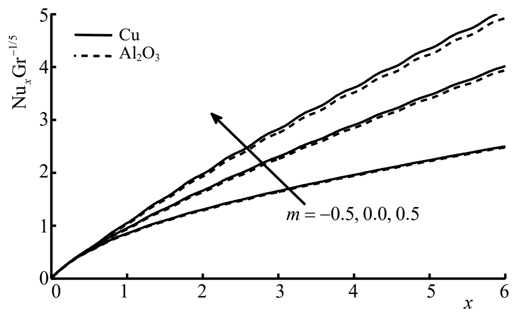

Figure 7. The variation of Nu$_x$Gr$^{-1/5}$ (local Nusselt number) against x, when ϕ = 0.2, Pr = 6.2 and α = 0.1 are fixed

Al_2O_3-water nanofluid because density and thermal conductivity of Cu-water nanofluid is larger than Al_2O_3-water nanofuid.

Figures 8 and 9 show the variation in C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ against amplitude of wavy surface, α, for pure water and both nanofluids. It is seen that as the amplitude of wavy plate increases C_{fx}Gr$^{1/5}$ and Nu$_x$Gr$^{-1/5}$ decrease in pure water and for both nanofluids. The reason for this decrease is that, with increase in amplitude the roughness of wavy surface enhances the viscous effects and hence causes a decrease in flow and heat transfer rates.

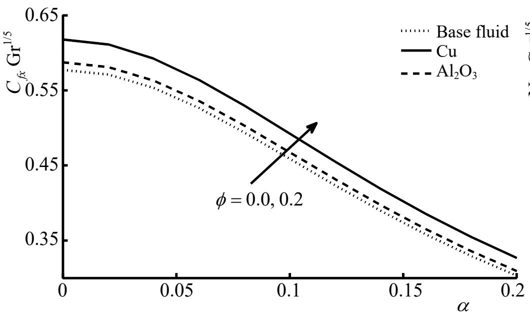

Figure 8. The variation of C_{fx}Gr$^{1/5}$ (local skin friction coefficient) amplitude α, when m = 1.0, Pr = 6.2 and x = 1.0 are fixed

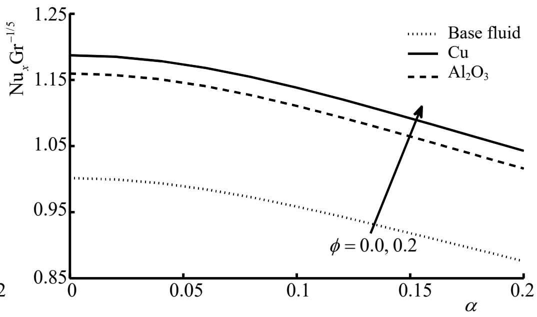

Figure 9. The variation of Nu$_x$Gr$^{-1/5}$ (local Nusselt number) against amplitude α, when Pr = 6.2, x = 1.0 and m = 1.0

Figures 10(a)-10(f) and figs. 11(a)-11(f) demonstrate the effect of nanoparticle volume fraction parameter, ϕ, on the streamlines and isothermal lines for fixed values of Pr = 6.2 and α = 0.1. Figures 10(a)-10(c) and figs. 11(a)-11(c) are plotted for uniform heat flux *i. e.* m = 0 and figs. 10(d)-10(f) and figs. 11(d)-11(f) are plotted for variable heat flux *i. e.* m = 0.5 for both pure water and nanofluids. In streamlines, the maximum values of ψ in uniform heat flux and variable heat flux are shown as ψ_{max} 4.0364 (ϕ = 0, pure water), 3.9899 (ϕ = 0.2, Al_2O_3), 3.359 (ϕ = 0.2, Cu), and 3.4865 (ϕ = 0, pure water), 3.4116 (ϕ = 0.2, Al_2O_3), 2.8959 (ϕ =0.2, Cu). It is observed that velocity decreases in nanofluids as compared to pure water, this is because due to the enrichment of ϕ viscosity of nanofluid increases. In addition to this in nanofluids, it is also observed that velocity in Cu-water is smaller than Al_2O_3-water due to large density of Cu nanoparticle. In case of variable heat flux, the values of ψ become smaller as compare to the values of uniform

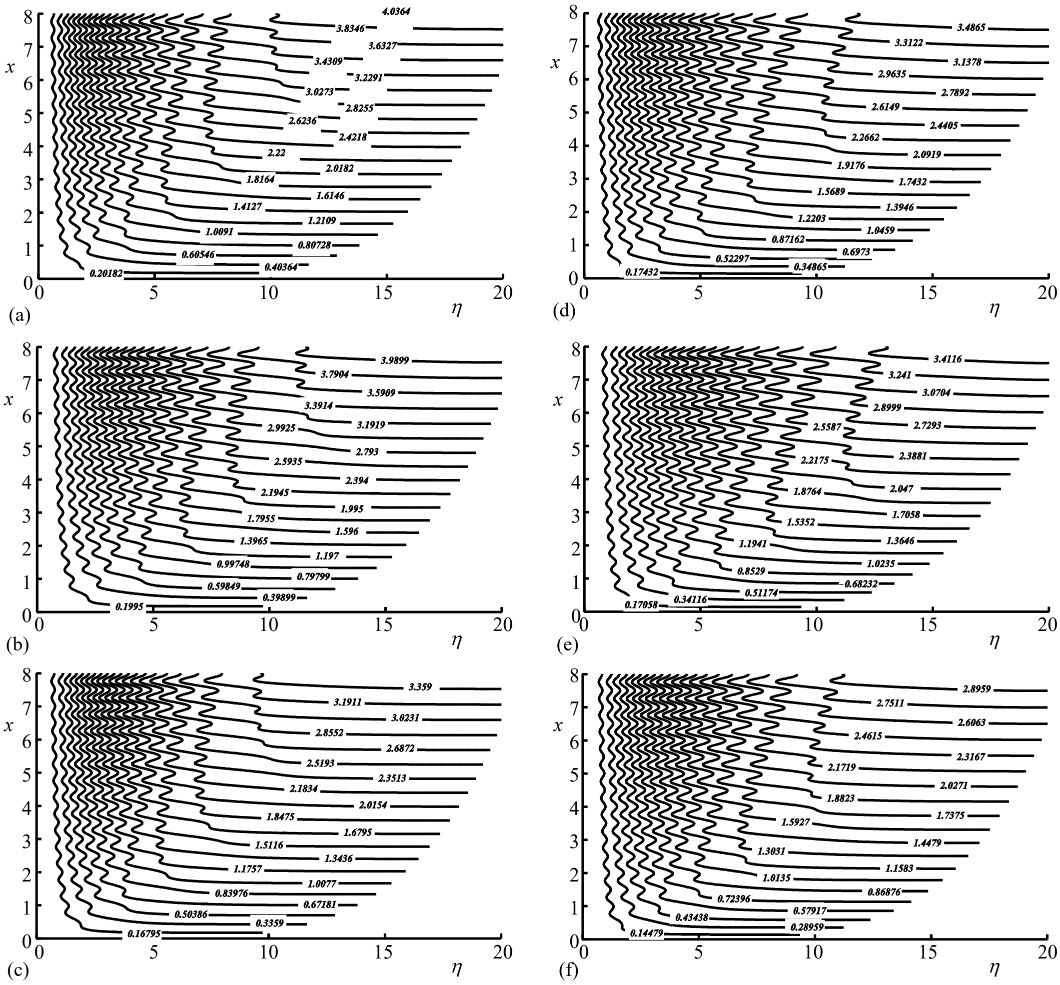

Figure 10(a)-10(f). Streamlines plot for m = 0.0 (a) ϕ = 0.0 (b) ϕ = 0.2 (Al_2O_3) (c) ϕ = 0.2 (Cu) and m = 0.5 (d) ϕ = 0.0 (e) ϕ = 0.2 (Al_2O_3) (f) ϕ = 0.2 (Cu) with Pr = 6.2 and α = 0.1

heat flux. Which is because velocity reduces in variable heat flux. In fig. 11(a)-11(f), it is seen that isothermal lines expands for both Al_2O_3-water and Cu-water nanofluids as compared to water base fluid. It reveals that temperature in nanofluids become higher due to larger thermal conductivity of nanofluids as pure water. It is also seen that isothermal lines contracts in case of variable heat flux *i. e.* m = 0.5 as compared to uniform heat flux *i. e.* m = 0. It reveals that temperature decreases within the boundary-layer in presence of variable heat flux.

Conclusions

In this article, we analyzed the way how one can enhance the rate of heat transfer in natural convection flow through a sinusoidal wavy plate. For this action, we incorporated two nanosize particles *i. e.* Al_2O_3 and Cu in a water base fluid. The solution of obtained equations is calculated numerically using Keller Box finite difference technique and effects of involving parameters namely, exponent of variable heat flux, m, amplitude of wavy surface, α, volume

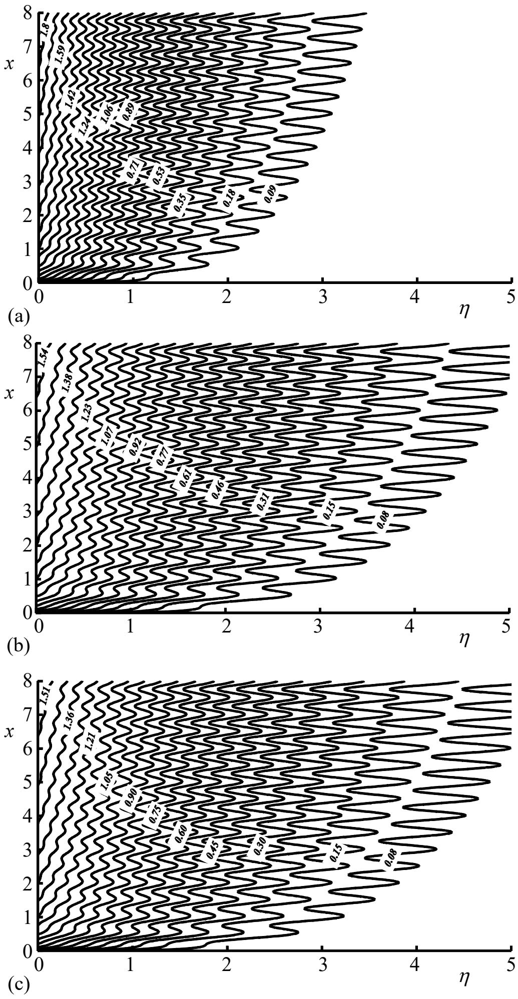

Figure 11(a)-11(f). Isothermal lines plot for m = 0.0 (a) ϕ = 0.0 (b) ϕ = 0.2 (Al_2O_3) (c) ϕ = 0.2 (Cu) and m = 0.5 (d) ϕ = 0.0 (e) ϕ = 0.2 (Al_2O_3) (f) ϕ = 0.2 (Cu) with Pr = 6.2 and α = 0.1

fraction parameter of nanoparticle, ϕ, and Prandtl number on $C_{fx}\text{Gr}^{1/5}$ and $\text{Nu}_x\text{Gr}^{-1/5}$ are examined through graphs. The core finding of the present analysis are mentioned as follows.

- The values of $C_{fx}\text{Gr}^{1/5}$ and $\text{Nu}_x\text{Gr}^{-1/5}$ or heat transfer rate for both Cu-water and Al_2O_3-water nanolfuids along wavy plate are higher than that of water base fluid.
- For increasing concentration of nanoparticle the values of $C_{fx}\text{Gr}^{1/5}$ and $\text{Nu}_x\text{Gr}^{-1/5}$ for Cu-water nanofluid is larger than Al_2O_3-water nanofluid for both flat and wavy plates but these values are smaller in case of wavy plate as compared to flat plate.
- The values of $C_{fx}\text{Gr}^{1/5}$ and $\text{Nu}_x\text{Gr}^{-1/5}$ reduce with increasing a amplitude of wavy sheet.
- With increasing the exponent of variable heat flux the values of $C_{fx}\text{Gr}^{1/5}$ decreases and $\text{Nu}_x\text{Gr}^{-1/5}$ increases.
- Velocity and temperature of nanofluid decreases and increases within the boundary-layer as compared to water base fluid.

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