MODIFIED KAWAHARA EQUATION WITHIN A FRACTIONAL DERIVATIVE WITH NON-SINGULAR KERNEL

by

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The article addresses a time-fractional modified Kawahara equation through a fractional derivative with exponential kernel. The Kawahara equation describes the generation of non-linear water-waves in the long-wavelength regime. The numerical solution of the fractional model of modified version of Kawahara equation is derived with the help of iterative scheme and the stability of applied technique is established. In order to demonstrate the usability and effectiveness of the new fractional derivative to describe water-waves in the long-wavelength regime, numerical results are presented graphically.

Key words: *fractional model of modified Kawahara equation, water-waves, Caputo-Fabrizio fractional derivative, iterative method*

Introduction, definitions, and preliminaries

Fractional derivatives and integrals have been gaining more and more interest of scientists due to their extensive applications in different directions of science, social science, engineering and finance [1-9] when the relaxation process have to accounted for. In this context, Atangana [10] analyzed the fractional non-linear Fisher's reaction-diffusion equation associated with Caputo-Fabrizio (CF) fractional derivative. Kumar *et al.* [11] studied the fractional non-linear shock wave equation by using homotopy analysis transform algorithm. In another work Bulut *et al.* [12] analyzed the non-linear fractional KdV-Burgers-Kuramoto equations with the help of modified trial equation method. Kumar *et al.* [13] investigated a fractional differential-difference equation occurring in nanotechnology and shown that fractional model describes the physical problem with high accuracy. Singh *et al.* [14] examined a Tricomi equation associated with local fractional derivative describing fractal transonic flow and obtained the non-differentiable solution of the problem. In an attempt Choudhary *et al.* [15] investigated fractional model of temperature distribution and heat flux in the semi infinite solid by using integral transform technique. Singh *et al.* [16] introduced a novel analytical technique for non-linear fractional differential equations and reported the numerical solution of coupled Burgers equations of arbitrary order. Heydari *et al.* [17] obtained the numerical solution of fractional optimal control problems with the help of the wavelets approach. In a recent investigation Atangana and Baleanu [18] proposed a novel fractional derivative and shown its efficiency to

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describe heat transfer. Gomez-Aguilar *et al.* [19] examined the electrical RLC circuit pertaining to fractional derivatives by using Liouville-Caputo operators. Coronel-Escamilla *et al.* [20] studied a fractional model of Euler-Lagrange and Hamilton equations containing fractional derivatives having exponential kernel. In view of great importance of fractional calculus, various approaches of fractional calculus have been discovered for example the Riemann-Liouville definition, the Caputo definition, *etc.*

In a recent work Caputo and Fabrizio [1] presented a new derivative of arbitrary order with exponential and non-singular kernel. The newly fractional is much better and efficient over the classical Caputo derivative is that by using new one the full outcome of the memory can be predicted. In this article, we use the newly fractional derivative approach suggested by CF to analyze a modified Kawahara equation and show that compare to ancient edition, the CF fractional derivative has additional stimulus effects.

Definition 1. Let us consider that $\varphi \in H^1(\alpha,\beta),\ \beta > \alpha,\ 0 \le \gamma \le 1$, then the CF derivative of fractional order discovered by Caputo and Fabrizio [1] is written in the following manner:

$$\mathrm{D}_t^\gamma\left[\varphi(t)\right] = \frac{M(\gamma)}{1-\gamma}\int_\alpha^t \varphi'(\tau)\exp\left[-\gamma\frac{t-\tau}{1-\gamma}\right]\mathrm{d}\tau \tag{1}$$

In this expression $M(\gamma)$ is indicating normalization function, which holds the property $M(0) = M(1) = 1$ [1].

In an attempt Losada Nieto [2] suggested the associate integral of the CF fractional derivative as presented in the following manner.

Definition 2. Let us consider that $\varphi(t)$ be a function, then the fractional integral operator of $\varphi(t)$ of order $\gamma, 0 < \gamma < 1$ is written [2]:

$$I_\gamma^t\left[\varphi(t)\right] = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\varphi(t) + \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t \varphi(s)\,\mathrm{d}s,\ \ t \ge 0 \tag{2}$$

It is to be worth noting that the CF derivative can easily be derived from the Cattaneo concept of the flux relaxation if the damping function is the Jeffrey memory kernel. This was demonstrated recently by Hristov for time-fractional [21] and space fractional [22] CF derivatives.

Fractional model of modified Kawahara equation

The Kawahara equation [23] is used to unfold the theory of the water-waves in the long-wave regime for modest values of surface tension, precisely for Weber numbers close to 1/3 [24, 25]. For such values of the Weber number the general model of long water-waves through the KdV equation is not adequate due to the cubic term in the linear modelling ends and fifth order dispersion proves applicable at leading order, $w(k) = k^5 + \lambda k^3$.

The Kawahara equation and its modified versions are intensively investigated, see [26-29] and the references therein and generally can be expressed:

$$v_t + v^2 v_x + \lambda v_{xxx} + \mu v_{xxxxx} = 0 \tag{3}$$

with the initial condition:

$$v(x,0) = v_0(x) \tag{4}$$

Replacing the time-derivative in eq. (3) by a derivative of fractional order termed in any sense (Riemann-Liouville, Caputo, *etc.*) we may transform eq. (3) to the modified Kawahara equation of fractional order. In the reference of the introductory analysis we consider the case when the time-fractional derivative is of CF sense, then the fractional modified Kawahara equation:

$$ {}^{\mathrm{CF}}_{0}\mathrm{D}_t^{\gamma} v(x,t) + v^2 v_x + \lambda v_{xxx} + \mu v_{xxxxx} = 0 \tag{5} $$

along with the initial condition (4).

Solution of fractional model of modified Kawahara equation by iterative approach

The solution approach of the time factional modified Kawahara eq. (5) addresses a numerical scheme. For this purpose initially the model of eq. (5) has to be transformed by the Laplace transform, namely:

$$ L[v(x,t)] = v(x,0)\frac{1}{p} - \frac{p+\gamma(1-p)}{M(\gamma)p} L\left[v^2 v_x + \lambda v_{xxx} + \mu v_{xxxxx}\right] \tag{6} $$

Now, applying the inverse of Laplace transform to eq. (6) we get:

$$ v(x,t) = v(x,0) - L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(v^2 v_x + \lambda v_{xxx} + \mu v_{xxxxx}\right)\right] \tag{7} $$

Therefore, the recursive formula can be expressed:

$$ v_0(x,t) = v(x,0) \tag{8} $$

and

$$ v_{n+1}(x,t) = v_n(x,t) - L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(v_n^2 \frac{\partial u_n}{\partial x} + \lambda \frac{\partial^3 v_n}{\partial x^3} + \mu \frac{\partial^5 v_n}{\partial x^5}\right)\right] \tag{9} $$

Finally, the solution of the time fractional modified Kawahara eq. (5) can be attained, that is:

$$ v(x,t) = \lim_{n\to\infty} v_n(x,t) \tag{10} $$

Stability analysis of iterative technique

As it is well known that the stability of iterative scheme plays a crucial role to find the solution of investigated equation. Thus, below we give a detailed proof of this issue.

It is assumed that $(X,\|\cdot\|)$ represents a Banach space and H indicates a self-map of X. We consider a specific recursive procedure $y_{n+1} = \varphi(H, y_n)$. It is considered that $G(H)$ be the fixed point set of H has at least one element and that y_n converges to a point $s \in G(H)$. If $\{x_n\} \subseteq X$ and it is defined that $e_n = \|x_{n+1} - \varphi(H, x_n)\|$. If the $\lim_{n\to\infty} \mathrm{e}^n = 0$ results to $\lim_{n\to\infty} x^n = s$, then the iteration scheme $y_{n+1} = \varphi(H, y_n)$ is known as H-Sable. Furthermore, it is assumed that the sequence $\{x_n\}$ has an upper boundary otherwise it can not be expected possibility of convergence. Whenever all these restrictions are fulfilled for $y_{n+1} = \varphi(H, y_n)$ which is said to be Picard's iteration, as a result the iteration will be H-Stable. Below we will present the result expressed by *Theorem 1*.

Before doing this issue we recall the following important result mentioned in [30].

We assume that $(X,\|\cdot\|)$ be a Banach space and H be a self-map of X that satisfies the following inequality:

$$\left\|H_x - H_y\right\| \le W\left\|x - H_x\right\| + w\left\|x - y\right\| \tag{11}$$

for all the values of x, y in X, where the values of W and w such that $0 \le W,\ 0 \le w \le 1$. It is assumed that H is Picard's H-Stable.

Now, let us consider the iterative formula connected to the time fractional modified Kawahara eq. (5) expressed in the form:

$$v_{n+1}(x,t) = v_n(x,t) - L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(\tilde{v}_n^2 \frac{\partial \tilde{v}_n}{\partial x} + \lambda \frac{\partial^3 v_n}{\partial x^3} + \mu \frac{\partial^5 v_n}{\partial x^5}\right)\right] \tag{12}$$

In eq. (12) the term $[p+\gamma(1-p)]/[M(\gamma)p]$ is the Lagrange's multiplier whereas $\tilde{v}_n$ denotes a restricted variation satisfying the condition $\delta \tilde{v}_n^2 (\partial \tilde{v}_n / \partial x) = 0$.

Next, we would like to prove the subsequent result expressed in the form of the theorem.

Theorem 1. Let us suppose that F be a self-map defined in the following manner:

$$F\left[v_n(x,t)\right] = v_{n+1}(x,t) = v_n(x,t) - L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(v_n^2 \frac{\partial v_n}{\partial x} + \lambda \frac{\partial^3 v_n}{\partial x^3} + \mu \frac{\partial^5 v_n}{\partial x^5}\right)\right] \tag{13}$$

then the iteration is F-Stable in $L^2(\alpha, \beta)$ if the following condition is satisfied:

$$\left[1 + \frac{1}{3}\rho_1 (A^2 + B^2 + AB)\psi_1(\eta) + \lambda \rho_2^3 \psi_2(\eta) + \mu \rho_3^5 \psi_3(\eta)\right] < 1$$

where ψ_1, ψ_2, and ψ_3 are functions arising from:

$$L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L(\bullet)\right]$$

Proof. Initially we will show that F has a fixed point. In order to establish this result, we compute the subsequent result for $(n,m) \in \mathrm{N} \times \mathrm{N}$.

$$\left\|F\left[v_n(x,t)\right] - F\left[v_m(x,t)\right]\right\| = \left\|v_n(x,t) - v_m(x,t)\right\| + \left\|L^{-1}\left\{\frac{p+\gamma(1-p)}{M(\gamma)p}\cdot\right.\right.$$

$$\left.\left.\cdot L\left[\frac{1}{3}\left(\frac{\partial(v_n^3)}{\partial x} - \frac{\partial(v_m^3)}{\partial x}\right) + \lambda \frac{\partial^3}{\partial x^3}(v_n - v_m) + \mu \frac{\partial^5}{\partial x^5}(v_n - v_m)\right]\right\}\right\| \tag{14}$$

Next with the help of the properties of norm, we get:

$$\left\|F\left[v_n(x,t)\right] - F\left[v_m(x,t)\right]\right\| \le \left\|v_n(x,t) - v_m(x,t)\right\| +$$

$$+L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(\left\|\frac{1}{3}\rho_1 (v_n^3 - v_m^3)\right\|\right)\right] + L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(\left\|\lambda \rho_2^3 (v_n - v_m)\right\|\right)\right] +$$

$$+L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p} L\left(\left\|\mu \rho_3^5 (v_n - v_m)\right\|\right)\right] \tag{15}$$

Since $v_n(x,t)$ and $v_m(x,t)$ are bounded functions so we are able to get two distinct constants A & $B>0$ *s. t.* for all the values of *t*:

$$\|v_n(x,t)\| \le A, \ \|v_m(x,t)\| \le B \tag{16}$$

Further, using the result of eq. (16) with respect to eq. (15) we have:

$$\|F[v_n(x,t)] - F[v_m(x,t)]\| \le \left[1+\frac{1}{3}\rho_1(A^2+B^2+AB)\psi_1(\eta)+\lambda\rho_2^3\psi_2(\eta)+\mu\rho_3^5\psi_3(\eta)\right]\cdot \cdot\|v_n(x,t)-v_m(x,t)\| \tag{17}$$

where ψ_1, ψ_2, and ψ_3 are functions arising from:

$$L^{-1}\left[\frac{p+\gamma(1-p)}{M(\gamma)p}L(\bullet)\right]$$

with

$$\left[1+\frac{1}{3}\rho_1(A^2+B^2+AB)\psi_1(\eta)+\lambda\rho_2^3\psi_2(\eta)+\mu\rho_3^5\psi_3(\eta)\right]<1$$

Thus, the non-linear *F*-self mapping attains a fixed point. Now, we shall prove that *F* fulfill the conditions with the inequality (11). Precisely, let eq. (17) be held, then we have:

$$w=0, \quad W=1+\frac{1}{3}\rho_1(A^2+B^2+AB)\psi_1(\eta)+\lambda\rho_2^3\psi_2(\eta)+\mu\rho_3^5\psi_3(\eta) \tag{18}$$

Therefore eq. (18) reveals that for the non-linear mapping *F* along with the inequality (11) holds. In addition, for the assumed non-linear mapping *F* along with all the conditions presented with inequality (11) satisfied, then *F* is Picard's *F*-Stable. It closes the proof and verifies the Theorem 1.

Numerical simulations

In the present section numerical simulations of the special solution of eq. (5) as function of the physical variables *t* and *x* for distinct values of γ at $\lambda=1$ and $\mu=-1$ are developed.

For the numerical computation the initial condition is taken:

$$v(x,0)=\frac{3\lambda}{\sqrt{-10\mu}}\operatorname{sech}^2(\omega x)$$

where $\omega=(1/2)(-\lambda/5\mu)^{1/2}$ is a constant. Solutions for particular values of γ are shown in figs. 1-4. From figs. 1-4, it can be noticed that the solutions of time-fractional modified Kawahara equation is in wave form and describes the displacements of water-waves in a very efficient manner. The time-fractional modified Kawahara equation reveals new characteristics for $\gamma=0.75$, $\gamma=0.50$, and $\gamma=0.25$ which were invisible in the integer-order version ($\gamma=1$). As it is shown in fig. 5 with the increase in γ, the displacement $v(x,t)$ of water-waves decreases but afterward its nature is opposite *i. e.* after some time with increase in γ, the displacement $v(x,t)$ of water-waves increases. The response displacement $v(x,t)$ of water-waves against the space variable *x* for different value of γ is presented in fig. 6. It can be observed from fig. 6 that the displacement $v(x,t)$ of water-waves increases with decreasing the value of γ but afterward its

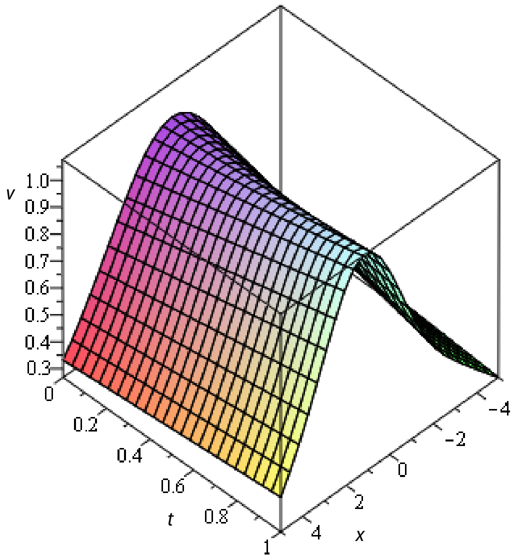

Figure 1. The surface of $v(x,t)$ w.r.t space x and time t are found at $\lambda = 1$, $\mu = -1$, and $\gamma = 1$

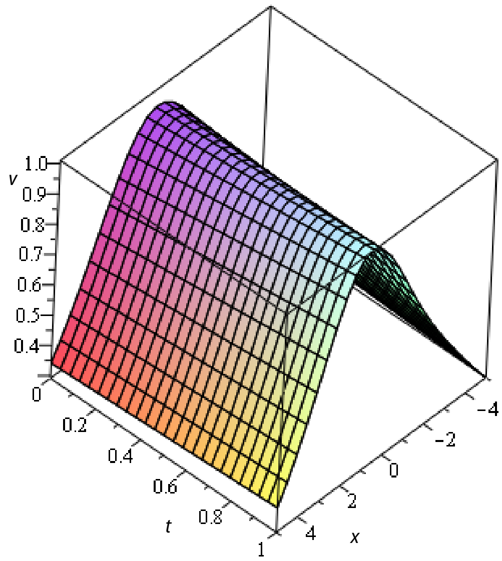

Figure 2. The surface of $v(x,t)$ w.r.t space x and time t are found at $\lambda = 1$, $\mu = -1$, and $\gamma = 0.75$

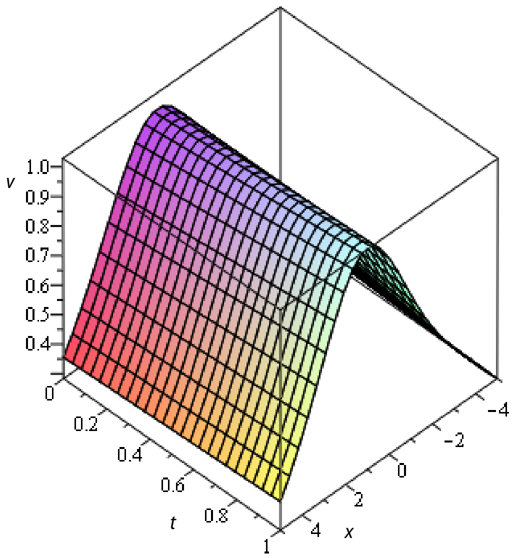

Figure 3. The nature of $v(x,t)$ w.r.t x and t are found at $\lambda = 1$, $\mu = -1$, and $\gamma = 0.50$

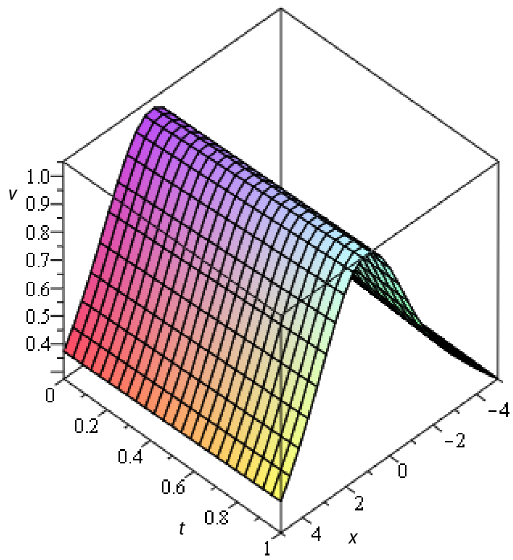

Figure 4. The response of $v(x,t)$ w.r.t space x and time t are found at $\lambda = 1$, $\mu = -1$, and $\gamma = 0.25$

nature is opposite *i. e.* after some distance with increase in γ, the displacement $v(x,t)$ of water-waves increases.

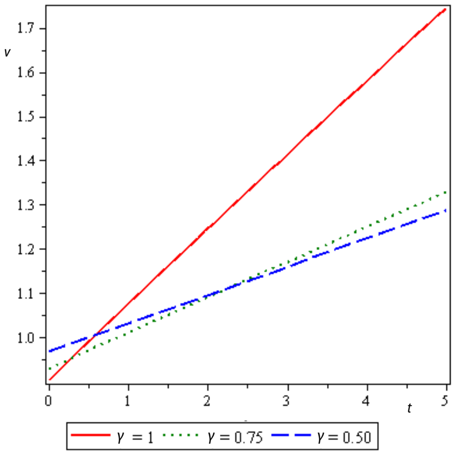

Figure. 5. The plots of $v(x,t)$ *vs.* t at $\lambda = 1$, $\mu = -1$, and $x = 1$ for different values of γ

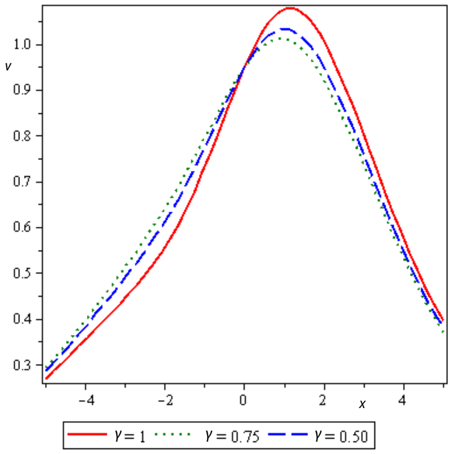

Figure 6. The plots of $v(x,t)$ *vs.* x at $\lambda = 1$, $\mu = -1$, and $t = 1$ for different values of γ

Conclusion

In this work, we have considered the fractional model of modified Kawahara equation associated with CF fractional derivative. Its numerical solution is obtained with the help of iterative scheme. In order to examine the stability of the iterative approach we employed the theory of F-stable mapping and the fixed-point approach. Some interesting numerical result for different values of γ at $\lambda = 1$ and $\mu = -1$ are obtained. The numerical results for fractional model of modified Kawahara equation shows that with increasing the order of time-fractional derivative the displacement of water-waves decreases but after some time with increasing the order of time-fractional derivative, the displacement of water-waves increases. The newly CF derivative has many useful and worth mentioning qualities for instance at distinct scales it can illustrate the matter diversities and configurations, where in local theories clearly it can not be controlled. The numerical results demonstrate that the new CF derivative can be employed to represent the real world problems.

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