AN ANALYTICAL SOLUTION FOR A FRACTIONAL HEAT-LIKE EQUATION WITH VARIABLE COEFFICIENTS

by

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The fractional power series method is used to solve a fractional heat-like equations with variable coefficients. The solution process is elucidated, and the results show that the method is simple but effective.

Key words: *fractional power series, fractional heat-like equations, variable coefficients, Caputo fractional derivative*

Introduction

The heat equation and its fractional generalization are used in many applications in science and engineering. There has been a growing interest to investigate the solutions and their properties for this equation for various reasons which include modeling of anomalous diffusive systems, description of fractional random walk, unification of diffusion phenomena. For more details see [1-7].

But fractional order differential equations are usually hard to solve analytically and exact solution are rather difficult to be obtained. So it is very important to find efficient methods for solving these equations.

Various researchers have introduced new methods in the literature. These methods include variational iteration method [1, 8-10], homotopy perturbation method [9], Daftardar--Gejji-Jafan method [4], Adomian decomposition method [3, 6, 11], and other methods [11-15].

In this paper, we will use fractional power series method (FPSM) [8, 12] to solve a fractional heat-like equations with variable coefficients [2, 15-18]. Compared to the previous--mentioned methods, the FPSM is more simple and effective.

Basic definitions of fractional calculus

Fractional calculus unifies and generalizes the notions of integer-order differentiation and n-fold integration. We give some basic definitions and properties of fractional calculus theory which shall be used in this paper. See [19] for detail of proof.

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

The Riemann-Liouville fractional integral operator is defined as follows.

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Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) \mathrm{d}s, \quad x > 0.$$

$$J^0 f(x) = f(x).$$

In this paper only real and positive values of α will be considered.

Properties of the operator J^α can be found in [19] and we mention only the following: for $\alpha, \beta \geq 0$, $x > 0$ and $\gamma > -1$:

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$$

$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} x^{\gamma+\alpha}$$

Definition 3. The fractional derivative of $f(x)$ in Riemann-Liouville sense is defined:

$$\mathrm{D}_*^\alpha f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f(s) \mathrm{d}s \tag{1}$$

for $m-1 < \alpha \leq m$, $m \in N^+$, $x > 0$, and $f \in C_{-1}^m$.

The Riemann-Liouville fractional derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall use a modified fractional differential D^0 proposed by Caputo in his work on the theory of viscoelasticity [19].

Definition 4. The fractional derivative of $f(x)$ in Caputo sense is defined:

$$\mathrm{D}^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s \tag{2}$$

for $m-1 < \alpha \leq m$, $m \in N^+$, $x > 0$ and $f \in C_{-1}^m$.

We recall here two of its basic properties [19]:

$$\mathrm{D}^\alpha J^\alpha f(x) = f(x)$$

$$J^\alpha \mathrm{D}^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0$$

Fractional power series (FPS) representation

Power series is a fundamental tool in the study of elementary functions. They have been widely used in computational science for easily obtaining an approximation of functions. In thermal physics and many other sciences this power expansion has allowed scientist to make an approximate study of many differential equations. Now, we will generalize some important definitions and theorems related with the classical power series into the fractional case in the sense of the Caputo definition [5, 12-14].

Definition 1. A power series representation of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots$$

where $0 \le m-1 < \alpha \le m$, $m \in N^+$ and $t \ge t_0$ is called a FPS about t_0, where t is a variable and c_n are the coefficients of the series.

Theorem 1. We have the following two cases for the $\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $t \ge 0$:

- If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges when $t = b > 0$ then it converges whenever $0 \le t < b$.
- If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ diverges when $t = d > 0$ then it diverges whenever $t > d$.

Theorem 2. For the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $t \ge 0$, there are only three possibilities.

- The series converges only when $t = 0$.
- The series converges for each $t \ge 0$.
- There is a positive real number R such that the series converges whenever $0 \le t < R$ and diverges whenever $t > R$.

Theorem 3. The series $\sum_{n=0}^{\infty} c_n t^{n}$, $-\infty < t < \infty$ has radius of convergence R if and only if the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$, $t \ge 0$ has radius of convergence $R^{1/\alpha}$.

The following property plays an important role in next section.

Theorem 4. Suppose that the FPS has $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ radius of convergence $R > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$ on $0 \le t < R$, then for $m-1 < \alpha \le m$ and $0 \le t < R$, we have:

$$\mathrm{D}^{\alpha} f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma[(n-1)\alpha+1]} t^{(n-1)\alpha} \tag{3}$$

Fractional heat-like equation

In this section we derive the main algorithms of the FPSM for fractional heat-like equations with variable coefficients.

Consider the fractional heat-like equations in the form:

$$\mathrm{D}_t^{\alpha} u = K(x) \mathrm{D}_x^{\beta} u \tag{4}$$

with the initial condition

$$u(x,0) = f(x) \tag{5}$$

where $0 < \alpha \le 1$ is a parameter describing the order of the Caputo fractional time derivative and $1 < \beta \le 2$ is a parameter describing the order of the Caputo fractional space derivative.

To apply FPSM, suppose that the solution of eq. (4) takes the form:

$$u(x,t) = \sum_{k=0}^{\infty} a_k(x) t^{\alpha k} = a_0(x) + a_1(x) t^{\alpha} + a_2(x) t^{2\alpha} + \cdots \tag{6}$$

The components $a_k(x)$ (k = 1, 2,…) will be determined recursively. Using eq. (5), we have $a_0(x) = f(x)$.

From *Theorem 4*, we obtain:

$$\mathrm{D}_x^\alpha u(x,t) = \sum_{k=1}^{\infty} \frac{a_k(x)\Gamma(\alpha k+1)}{\Gamma[\alpha(k-1)+1]} t^{\alpha(k-1)} \tag{7}$$

On the other hand:

$$\mathrm{D}_x^\beta u = \mathrm{D}_x^\beta a_0(x) + t^\alpha \mathrm{D}_x^\beta a_1(x) + t^{2\alpha} \mathrm{D}_x^\beta a_2(x) + \cdots \tag{8}$$

Substituting eqs. (7) and (8) into eq. (4), and comparing the coefficients of t^α in the both side, we get:

$$a_k(x) = \frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)} K(x) \mathrm{D}_x^\beta a_{k-1}(x), \quad k = 1, 2, \cdots \tag{9}$$

Thus we obtain the solution:

$$u(x,t) = f(x) + \sum_{k=1}^{\infty} \frac{\Gamma[\alpha(k-1)+1]}{\Gamma(\alpha k+1)} K(x) \mathrm{D}_x^\beta a_{k-1}(x) t^{\alpha k} \tag{10}$$

It is worth noting that the zeroth component $a_0(x)$ is defined than the remaining components $a_k(x)$ (k = 1, 2,…) can be completely determined such that each terms are determined by using the previous terms, and the series solutions thus entirely determined by eq. (10).

However, in many cases the exact solution in a closed form may be obtained. Moreover, the series solutions are generally converge very rapidly. The convergence of the FPS have investigated by several authors [5-7]. To give a clear overview of the discussion presented previously, the following examples will be studied.

Consider the following fractional heat-like problem:

$$\mathrm{D}_t^\alpha u = \frac{1}{2} x^2 \mathrm{D}_x^\beta u \tag{11}$$

with the initial condition $u(x, 0) = x^2$.

We suppose that the solution of eq. (11) takes the form:

$$u(t) = a_0(x) + a_1(x)t^\alpha + a_2(x)t^{2\alpha} + a_3(x)t^{3\alpha} + \cdots \tag{12}$$

Using the initial condition $u(x, 0) = x^2$, we choose $a_0 = x^2$. Next we determine the $a_k(x)$, (k = 1, 2, …).

On the one hand:

$$\mathrm{D}_x^\alpha u(x,t) = \sum_{k=1}^{\infty} \frac{a_k(x)\Gamma(\alpha k+1)}{\Gamma[\alpha(k-1)+1]} t^{\alpha(k-1)} \tag{13}$$

On the other hand:

$$\mathrm{D}_x^\beta u = \mathrm{D}_x^\beta a_0(x) + t^\alpha \mathrm{D}_x^\beta a_1(x) + t^{2\alpha} \mathrm{D}_x^\beta a_2(x) + \cdots \tag{14}$$

Substitute eqs. (13) and (14) into eq. (11), and the equating of the coefficients of t^{ak} in both side, leads to the following:

$$a_1(x) = \frac{x^{4-\beta}}{\Gamma(\alpha+1)\Gamma(3-\beta)}$$

$$a_2(x) = \frac{\Gamma(5-\beta)x^{6-2\beta}}{2\Gamma(2\alpha+1)\Gamma(3-\beta)\Gamma(5-2\beta)}$$

$$a_3(x) = \frac{\Gamma(5-\beta)\Gamma(7-2\beta)x^{8-3\beta}}{4\Gamma(3\alpha+1)\Gamma(3-\beta)\Gamma(5-2\beta)\Gamma(7-3\beta)}$$

and so on.

Thus the solution can be obtained. For example, when $\alpha = 1$, $\beta = 2$, we have the solution:

$$\begin{aligned} u &= x^2 + x^2 t + \frac{1}{2}x^2t^2 + \frac{1}{6}x^2t^3 + \cdots \\ &= x^2 e^t \end{aligned}$$

which agrees with the results of this problem obtained in [2].

Conclusion

We have used FPSM to solve the fractional heat-like equations with variable coefficients. Compared to the other method, the FPSM is more simple, direct, and effective.

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