ON VARIATIONAL ITERATION METHOD FOR FRACTIONAL CALCULUS

by

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Modification of the Das' variational iteration method for fractional differential equations is discussed, and its main shortcoming involved in the solution process is pointed out and overcome by using fractional power series. The suggested computational procedure is simple and reliable for fractional calculus.

Key words: *variational iteration method, fractional power series, fractional equation*

Introduction

Fractional differential equations are usually arising from mathematical modeling of many physical phenomena, especially in thermal science. In most cases, it is very difficult to achieve exact solutions. The variational iteration method (VIM) is the most potential candidate for fractional calculus [1-8], however, much attention has to be paid on differential fractional derivatives appeared in open literature.

As an example, we consider the following initial value problem:

$$\begin{cases} \dfrac{\partial^{1/2}u(x,t)}{\partial t^{1/2}} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial}{\partial x}\left[xu(x,t)\right], \\ u(x,0) = f(x). \end{cases} \tag{1}$$

where $u(x, t)$ represents temperature, $f(x)$ – a known function, and $\partial^{1/2}/\partial t^{1/2}(\cdot)$ – the Caputo derivative of order 1/2.

Equation (1) describes a heat conduction problem of porous media with a variable coefficient of thermal conductivity.

Das [9] first rewrite the problem eq. (1) as:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^{1/2}}{\partial t^{1/2}}\dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial^{1/2}}{\partial t^{1/2}}\dfrac{\partial}{\partial x}\left[xu(x,t)\right], \\ u(x,0) = f(x). \end{cases} \tag{2}$$

and obtained the following iteration formulas:

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$$u_0(x,t) = f(x),$$
$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - \frac{\partial^{1/2}}{\partial \xi^{1/2}} \frac{\partial}{\partial x}[x u_n(x,\xi)] \right\} \mathrm{d}\xi \tag{3}$$

where $n = 0, 1, 2 \ldots$

Finally the exact solution is obtained by:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

However, we have noted that the problems (1) and (2) are not equivalent, and by (3), we can only obtain $u_n(x, t) \equiv u_0(x, t)$ ($n = 1, 2, \ldots$), which is in contradiction to Das's illustrative examples [9].

In fact, if let $f(x) = 1$, then $u(x, t) = 1$ is the solution of eq. (2). But the solution is not solution of eq. (1). The example shows that the iteration of eqs. (3) produce solutions which do not satisfy the original differential equation. Here in the Section *Solution of the problem*, we will give the corresponding iteration formulas. Moreover, by using fractional power series [10], we present a simple computational framework for the construction of analytical solutions to the problem (1).

Definitions and properties

In this section, basic definitions and properties of the fractional calculus theory are introduced. For more detail, see [11].

Definition 1. Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of the function $u(x, t)$ with respect to t is defined:

$$J_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,\tau) \mathrm{d}\tau, \qquad t > 0,$$
$$J_t^0 u(x,t) = u(x,t) \tag{4}$$

The operator J_t^α satisfies the following properties, for $\alpha, \beta \geq 0$, and $\gamma \geq -1$:

$$J_t^\alpha J_t^\beta u(x,t) = J_t^{\alpha+\beta} u(x,t) \tag{5}$$

$$J_t^\alpha J_t^\beta u(x,t) = J_t^\beta J_t^\alpha u(x,t) \tag{6}$$

$$J_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha} \tag{7}$$

Definition 2. The fractional derivative of $u(x, t)$ with respect to t in the Riemann-Liouville sense is defined:

$$^{*}\mathrm{D}_t^\alpha u(x,t) = \frac{\partial^m}{\partial t^m} J_t^{m-\alpha} u(x,t) \tag{8}$$

where $m \in N$ and satisfies the relation $m - 1 < \alpha \leq m$.

According to the definition, one has:

$$ {}^{*}\mathrm{D}_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} \tag{9} $$

where, $\alpha \geq 0$, $t > 0$, and $\gamma \geq 1$.

Definition 3. The fractional derivative of $u(x, t)$ with respect to t in the Caputo sense is defined:

$$ \mathrm{D}_t^{\alpha} u(x,t) = J_t^{m-\alpha} \frac{\partial^m}{\partial t^m} u(x,t) \tag{10} $$

for $m - 1 < \alpha \leq m$, $m \in N$, and $x > 0$.

We recall here three of its basic properties:

$$ \mathrm{D}_t^{\alpha} J_t^{\alpha} u(x,t) = u(x,t) \tag{11} $$

$$ J_t^{\alpha} \mathrm{D}_t^{\alpha} u(x,t) = u(x,t) - \sum_{k=0}^{m-1} u_t^{(k)}(x,0^+) \frac{t^k}{k!}, \qquad t > 0 \tag{12} $$

We recall here three of its basic preperties:

$$ \mathrm{D}_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} \tag{13} $$

where $\gamma > m - 1$.

Solution of the problem (1)

We rewrite the problem (1) in the form:

$$ \begin{cases} \mathrm{D}_t^{1/2} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial}{\partial x^2}\left[x u(x,t)\right], \\ u(x,0) = f(x). \end{cases} \tag{14} $$

Using eqs. (5)-(8) and (10), we have:

$$ \begin{aligned} {}^{*}\mathrm{D}_t^{1/2} \mathrm{D}_t^{1/2} u(x,t) &= \frac{\partial}{\partial t} J_t^{1/2} J_t^{1/2} \frac{\partial}{\partial t} u(x,t) \\ &= \frac{\partial}{\partial t} J_t \frac{\partial}{\partial t} u(x,t) \\ &= \frac{\partial}{\partial t} u(x,t) \end{aligned} $$

and thus, the problem (1) is equivalent to the following problem:

$$ \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = {}^{*}\mathrm{D}_t^{1/2} \dfrac{\partial^2 u(x,t)}{\partial x^2} + {}^{*}\mathrm{D}_t^{1/2} \dfrac{\partial}{\partial x}\left[x u(x,t)\right], \\ u(x,0) = f(x). \end{cases} \tag{15} $$

According to the VIM, we can obtain the following iteration procedures:

$$u_0(x,t) = f(x),$$
$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\xi)}{\partial \xi} - {}^*\mathrm{D}_\xi^{1/2} \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - {}^*\mathrm{D}_\xi^{1/2} \frac{\partial}{\partial x}\left[x u_n(x,\xi) \right] \right\} \mathrm{d}\xi \tag{16}$$

where $n = 0, 1, 2, \ldots$. Consequently, the exact solution may be obtained by using:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

Remark. We remind the reader that *the symbol* ${}^*\mathrm{D}^{1/2}$ in eq. (16) represents the Riemann-Liouville fractional derivative. Actually, in *Examples 1-3* given by Das [9], author found the solutions by using iteration procedures, eq. (16).

Next we use fractional power series to solve the problem (1). Suppose that the solution takes the form:

$$u(x,t) = \sum_{n=0}^{\infty} a_n(x) t^{\frac{n}{2}} \tag{17}$$

From initial condition, we can obtain:

$$a_0(x) = f(x) \tag{18}$$

By (13), we get:

$$\mathrm{D}_t^{1/2} u(x,t) = \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} a_n(x) t^{\frac{n-1}{2}} \tag{19}$$

On the other hand:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{n=0}^{\infty} a_n''(x) t^{\frac{n}{2}} \tag{20}$$

$$\frac{\partial\left[x u(x,t) \right]}{\partial x} = \sum_{n=0}^{\infty} \left[x a_n'(x) + a_n(x) \right] t^{\frac{n}{2}} \tag{21}$$

Substitute eqs. (19), (20), and (21) into eq. (14) yields that:

$$\sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} a_n(x) t^{\frac{n-1}{2}} = \sum_{n=0}^{\infty} \left[a_n''(x) + x a_n'(x) + a_n(x) \right] t^{\frac{n}{2}} \tag{22}$$

By comparing the coefficients of $t^{n/2}$ in both sides of eq. (22), we get:

$$a_{n+1}(x) = \frac{\Gamma\left(\dfrac{n+2}{2}\right)}{\Gamma\left(\dfrac{n+3}{2}\right)}\left[a_n''(x) + x a_n'(x) + a_n(x)\right] \tag{23}$$

where $n = 0, 1, 2, \ldots$. Thus, the solution of eq. (1) can be constructed by eqs. (18) and (23).

Example 1. Consider the following problem:

$$\begin{cases} \mathrm{D}_t^{1/2} u(x,t) = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \dfrac{\partial}{\partial x}\left[x u(x,t)\right], \\ u(x,0) = 1. \end{cases} \tag{24}$$

Using eqs. (18) and (23), then:

$$a_0(x) = 1, \qquad a_1(x) = \frac{1}{\Gamma\frac{3}{2}}, \qquad a_2(x) = 1, \qquad a_3(x) = \frac{1}{\Gamma\frac{5}{2}}$$

and so on. The exact solution is:

$$u(x,t) = 1 + \frac{t^{1/2}}{\Gamma\frac{3}{2}} + t + \frac{t^{3/2}}{\Gamma\frac{5}{2}} + \ldots = \sum_{r=0}^{\infty} \frac{t^{r/2}}{\Gamma\left(\dfrac{r}{2}+1\right)} = E_{1/2}(\sqrt{t})$$

where

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \; (\alpha > 0)$$

is Mittag-Leffler function in one parameter. The previous result is in agreement with [9].

Example 2. Let us consider $u(x, 0) = \sin x$, then:

$$a_0(x) = \sin x, \qquad a_1(x) = \frac{1}{\Gamma\frac{3}{2}} x \sin x, \qquad a_2(x) = x^2 \cos x - 2\sin x$$

and so on. Thus the solution is:

$$u(x,t) = \sin x + \frac{t^{1/2}}{\Gamma\frac{3}{2}} x \cos x + t(x^2 \cos x - 2\sin x) + \cdots$$

Conclusion

In view of the different definitions of fractional operators, it seems to be a source of much confusion present in the literature. In the note, we have corrected Das's [9] results, and by using fractional power series we present a simple computational procedure for solving the problem of eq. (1). The presented algorithm is convenient for solving the problem.

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