

HYBRIDIZATION OF HOMOTOPY PERTURBATION METHOD AND LAPLACE TRANSFORMATION FOR THE PARTIAL DIFFERENTIAL EQUATIONS

by

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Homotopy perturbation method is combined with Laplace transformation to obtain approximate analytical solutions of non-linear differential equations. An example is given to elucidate the solution process and confirm reliability of the method. The result indicates superiority of the method over the conventional homotopy perturbation method due its flexibility in choosing its initial approximation.

Key words: *homotopy perturbation method, Laplace transformation, He-Laplace method, non-linear differential equations*

Introduction

The last two decades have experienced a rapid development of non-linear sciences arising in thermal science and other fields as well. Recently, several analytical methods were developed in order to find solutions of non-linear partial differential equations (PDE), for example, the homotopy perturbation method (HPM) [1, 2], the variational iteration method [3-5], the exp-function method [6-8], the homotopy analysis method [9, 10], and others [11, 12].

The classic approach by the perturbation method is still widely used for weakly non-linear equations and appeared in many textbooks. The drawback of the perturbation method is that their choice of a small parameter seems to be an art rather than a solution procedure, an inappropriate choice of such parameter leads to an inaccurate result or even a wrong one. The shortcoming can be easily solved by HPM, which was proposed in later 1990s [1, 2], and the method has been developed into a mature tool for almost all kinds of non-linear equations.

Laplace transform is accessible to all students, but it is suitable only for linear equations. Gondal and Khan [13] first coupled HPM with the Laplace transform, and the coupled technology is widely used in non-linear differential equations [14] and fractional calculus [11], and the method is generally called as He-Laplace method [15].

He-Laplace method

To illustrate the basic idea of this method, we consider the general form of 1-D non-homogeneous PDE with a variable coefficient of the form:

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$$\frac{\partial u}{\partial t} = \mu(x) \frac{\partial^2 u}{\partial x^2} + \varphi(x, t) \quad (1)$$

and

$$\frac{\partial^2 u}{\partial t^2} = \mu(x) \frac{\partial^2 u}{\partial x^2} + \varphi(x, t) \quad (2)$$

subject to the boundary conditions:

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t) \quad (3)$$

and the initial condition:

$$u(x, 0) = f(x) \quad (4)$$

The methodology consists of applying transform on both sides of eqs. (1) and (3) and in view of the initial condition [13-15]:

$$\frac{d^2 \bar{U}}{dx^2} - \frac{s \bar{u}(x, s)}{\mu(x)} + \frac{\bar{\varphi}(x, s) + f(x)}{\mu(x)} = 0 \quad (5)$$

$$\bar{u}(0, s) = g_0(s), \quad \bar{u}(1, s) = g_1(s) \quad (6)$$

which is second order boundary value problem. According to HPM [1, 2], we construct a homotopy in the form:

$$H(v, p) = (1-p) \left[\frac{d^2 v}{dx^2} - \frac{d^2 \bar{u}_0}{dx^2} \right] + p \left[\frac{d^2 v}{dx^2} - \frac{sv}{\mu(x)} + \frac{\bar{\varphi}(x, s) + f(x)}{\mu(x)} \right] = 0 \quad (7)$$

where \bar{u}_0 is the arbitrary function that satisfies boundary conditions eq. (6), therefore:

$$v(x, s) = \sum_{i=0}^8 p^i v_i(x, s) = v_0(x, s) + p^1 v_1(x, s) + p^2 v_2(x, s) + \dots \quad (8)$$

Applying the inverse Laplace transform on both sides of eq. (8), we get:

$$v(x, t) = \sum_{i=0}^8 p^i v_i(x, s) = v_0(x, t) + p^1 v_1(x, t) + p^2 v_2(x, t) + \dots \quad (9)$$

Setting $p = 1$ gives the approximate solution of eq. (9):

$$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + \dots \quad (10)$$

Implementation

In order to ascertain the wider applicability of the method we consider a non-linear differential equation of the form (11) as an example:

$$\frac{dU}{dt} = \frac{d^2 U}{dx^2} + (ae^{-t} + e^{-at}) \quad (11)$$

With the initial condition:

$$U(x, 0) = \sin(\pi x) \quad (12)$$

and boundary

$$U(0, t) = 0, \quad U_x = \pi(e^{-t} + e^{-at}) \quad (13)$$

By applying the method subject to the initial condition, we have:

$$\frac{d^2\bar{U}}{dx^2} + (\pi^2 - 1 - a - s)\bar{U} + 2\sin(\pi x) + \frac{a}{s+1} + \frac{1}{s+a} = 0 \quad (14)$$

$$\bar{U}(0, s) = 0, \quad \bar{U}_x(0, s) = \pi \left(\frac{1}{s+1} + \frac{1}{s+a} \right) \quad (15)$$

To solve eq. (14) by means of HPM, a homotopy equation can easily be constructed:

$$H(V, P) = (1 - P) \left[\frac{d^2V}{dx^2} - \frac{d^2\bar{U}_0}{dx^2} \right] + P \left[\frac{d^2V}{dx^2} + (\pi^2 - 1 - a - s)V + 2\sin(\pi x) + \frac{a}{s+1} + \frac{1}{s+a} \right] \quad (16)$$

We assume that the solution to eq. (16) may be written as a power series in P :

$$V(x, s) = \sum_{i=0}^{\infty} P^i V_i(x, s) = V_0(x, s) + P^1 V_1(x, s) + P^2 V_2(x, s) + \dots \quad (17)$$

Substituting eq. (17) into eq. (16), and equating the terms with the identical powers of p , we have:

$$\left\{ \begin{array}{l} p^0: \frac{d^2V}{dx^2} - \frac{d^2\bar{U}_0}{dx^2} = 0, \quad V_0(0, s) = 0, \quad V_{0x}(0, s) = \pi \left(\frac{1}{s+1} + \frac{1}{s+a} \right) \\ p^1: \frac{d^2V_1}{dx^2} + V_0(\pi^2 - 1 - a - s) + 2\sin \pi x(1 - \pi^2) + \frac{a}{s+1} + \frac{1}{s+a} = 0' \\ \quad V_1(0, s) = 0, \quad V_{1x}(0, s) = 0 \\ p^2: \frac{d^2V_2}{dx^2} + (\pi^2 - 1 - a - s)V_2 = 0, \quad V_2(0, s) = 0, \quad V_{2x}(0, s) = 0 \\ p^3: \frac{d^2V_3}{dx^2} + (\pi^2 - 1 - a - s)V_3 = 0, \quad V_3(0, s) = 0, \quad V_{3x}(0, s) = 0 \end{array} \right. \quad (18)$$

The initial approximation $V_0(x, s)$ or $\bar{U}_0(x, s)$ can freely be chosen, and we can set:

$$V_0(x, s) = \bar{U}_0(x, s) = \frac{\sin \pi x}{s+1} + \frac{\sin \pi x}{s+a} \quad (19)$$

which satisfies the boundary conditions (15).

Substituting eq. (19) into eq. (18), the first component of the homotopy perturbation solution for eq. (14) are derived:

$$\begin{aligned} v_1(x, s) = & (\pi^2 - 1 - a - s) \left[\frac{\sin \pi x}{\pi^4} \left(\frac{1}{s+1} + \frac{1}{s+a} \right) (\pi^2 - 1 - a - s) + \right. \\ & \left. + \frac{\sin \pi x}{\pi^2} (1 - \pi^2) + \frac{ax^2}{24(s+1)} + \frac{x^4}{24(s+a)} \right] + x(\pi^2 - 1 - a - s) \cdot \\ & \cdot \left[-\frac{1}{\pi^2} \left(\frac{1}{s+1} + \frac{1}{s+a} \right) (\pi^2 - 1 - a - s) - \frac{1}{\pi^2} (1 - \pi^2) \right] \end{aligned} \quad (20)$$

By taking the inverse Laplace of eq. (20) yields: $v_0(x, t) = e^{-t} \sin \pi x + e^{-at} \sin \pi x$:

$$v_1(x, y) = \frac{\sin \pi x}{\pi^2} (1 - \pi^2) - \frac{ax^2}{2} e^{-t} - \frac{x^2}{2} e^{-at} + \frac{\sin \pi x}{\pi^2} (\pi^2 - 2 - a) e^{-t} + \frac{\sin \pi x}{\pi^2} (\pi^2 - 1) e^{-at}$$

which is the exact solution.

Conclusion

In this work we employed He-Laplace method to find the exact solutions of some non-linear PDE. The method is very easy to implement due to its flexibility in freely chosen its initial approximation.

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