NUMERICAL METHOD FOR SINGULARLY PERTURBED DELAY PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

Yulan WANG**, Dan TIAN, and Zhiyuan LI

Department of Mathematics, Inner Mongolia University of Technology, Hohhot, China

Original scientific paper
http://doi.org/10.2298/TSCI160615040W

The barycentric interpolation collocation method is discussed in this paper, which is not valid for singularly perturbed delay partial differential equations. A modified version is proposed to overcome this disadvantage. Two numerical examples are provided to show the effectiveness of the present method.

Key words: *barycentric interpolation, singularly perturbed, delay parameter, Chebyshev nodes, Taylor's series expansion*

Introduction

Singularly perturbed delay partial differential equations arise from thermal science and mechanics systems which are characterized by both spatial and temporal variables, and exhibit various spatio-temporal patterns and provide more realistic models for thermal science where time-lag or after-effect has to be considered. A characteristic example is [1]:

$$\frac{\partial u_\varepsilon}{\partial t} = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2} + v\{g[u_\varepsilon(x, t-\tau)]\} \frac{\partial u_\varepsilon}{\partial x} + c\{f[u_\varepsilon(x, t-\tau)] - u_\varepsilon(x, t)\} \tag{1}$$

which models a furnace used to process a metal sheet. Here u_ε is the temperature distribution in a metal sheet, moving at a velocity, v, and heated by a source and specified by the function, f. Both v and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length. When $\tau = 0$, eq. (1) becomes a thermal problem without time delay.

When we select $D = (0, 1) \times (0, T)$, the problem considered is the following singularly perturbed delay parabolic equation with Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x, t-1) = f(x,t), (x,t) \in [0,1]\times[0,T] \\ u(x,t) = \psi(x,t), (x,t) \in [0,1]\times[-\gamma, 0] \\ u(0,t) = \varphi_T(t), t \in [0,T] \\ u(1,t) = \varphi_R(t), t \in [0,T] \end{cases} \tag{2}$$

where $0 < \varepsilon \ll 1$ is singular perturbed parameter, $f(x, t)$, $\psi(x, t)$, $\varphi_T(t)$, and $\varphi_R(t)$ are sufficiently smooth and bounded functions. The terminal time, T, is assumed to satisfy the condition

* Corresponding author, e-mail: wylnei@163.com

$T = K_T$, where K is a positive integer. Under the previous assumptions and conditions, problem (2) with the initial data and boundary conditions has a unique solution [1].

There are many methods to solve this problem [2-9], for example, the variational iteration method, the homotopy perturbation method, and others. In the past, the barycentric interpolation collocation method (BICM) has been presented and applied to many fields [2, 3]. However, the direct use of the method can not solve singularly perturbed delay partial differential equations, if ignore the delay parameter, it can not always get good result. For this kind of singularly perturbed delay partial differential equations, based on barycentric interpolation collocation method, by Taylor's series expansion, we give a modified BICM to solve them. Two numerical examples are given to demonstrate the efficiency of the present method.

Modified BICM

Expanding the delay term $u(x, t-\delta)$ around x by Taylor's series expansion, we obtain $u(x, t-\delta) \approx u(x, t) - \delta[\partial u(x, t)/\partial t]$, and eq. (2) can be approximated by the following singularly perturbed problem:

$$\begin{cases} Lu = [1-a(x)]\dfrac{\partial u(x,t)}{\partial t} - \varepsilon\dfrac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t) = f(x,t),\ (x,t)\in[0,1]\times[0,T] \\ u(x,t) = \psi(x,t),\ (x,t)\in[0,1]\times[-\tau,0] \\ u(0,t) = \varphi_T(t),\ t\in[0,T] \\ u(1,t) = \varphi_R(t),\ t\in[0,T] \end{cases} \tag{3}$$

The differential matrix of barycentric interpolation is [2]:

$$D_{ij}^{(1)} = L_j'(x_i),\ D_{ij}^{(2)} = L_j''(x_i) \tag{4}$$

$$\begin{cases} D_{ij}^{(m)} = m\left[D_{ii}^{(m-1)}D_{ij}^{(1)} - \dfrac{D_{ij}^{(m-1)}}{x_i - x_j}\right],\ i\neq j \\ D_{ii}^{(m)} = -\displaystyle\sum_{j=1, j\neq i}^{n} D_{ij}^{(m)} \end{cases} \tag{5}$$

In view of eq. (3), let interval [0, 1] be dispersed as $0 = x_1 < x_2 < \ldots < x_n, = 1$, interval [0, T] dispersed as $0 = t_1 < t_2 < \ldots < t_n = T$, let $u_1, u_2, \ldots u_n$ as the values of function $u(x)$ at disperse nodes $x_1, x_2, \ldots x_n$, respectively. The barycentric interpolation collocation is adopted to obtain an approximate solution of $u(x, t)$ in the form:

$$u(x,t) = \sum_{j=1}^{n} L_j(x)u_j \tag{6}$$

where $u_i(t)$ is expressed:

$$u_i(t) = \sum_{k=1}^{n} L_k(t)u_{ik} \tag{7}$$

By the assumption given in eqs. (6) and (7), we can obtain a matrix equation in the form $LU = F$, from eq. (3), where:

$$L = E(I_n \otimes D^{(1)}) - \varepsilon(C^{(2)} \otimes I_n) + A$$

$$U = [u_1, u_2, \ldots u_n]$$

$$A = \mathrm{diag}[A_i]$$

$$I = I_n \otimes I_n$$

$$E = I - \mathrm{A}$$

$$F = [f_1, f_2, \ldots f_n]^T$$

$$D^{(m)} = [D_{kj}^{(m)}(t)]_{n \times n}$$

$$C^{(m)} = [C_{kj}^{(m)}(t)]_{n \times n}, \qquad k,\ j = 1, 2, \ldots n$$

Numerical experiment

In this section, two numerical examples are studied to demonstrate the accuracy of the present method.

Example 1. Consider the following equation [4, 5]:

$$\begin{cases} \dfrac{\partial u(x, t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x, t)}{\partial x^2} = -2\mathrm{e}^{-1} u(x, t-1), (x, t) \in (0, 1) \times (0, 2] \\ u(x, t) = \mathrm{e}^{-\left(t + \frac{x}{\sqrt{\varepsilon}}\right)}, (x, t) \in [0, 1] \times [-1, 0] \\ u(0, t) = \mathrm{e}^{-t}, t \in [0, 2] \\ u(1, t) = \mathrm{e}^{-t - \frac{1}{\sqrt{\varepsilon}}}, t \in [0, 2] \end{cases}$$

The exact solution is:

$$u_T(x, t) = \mathrm{e}^{-\left(t + \frac{x}{\sqrt{\varepsilon}}\right)}$$

the error compared with fitted difference method in classical uniform meshes (CUM) and in fitted piecewise uniform meshes (FPUM) are shown in tab. 1.

Table 1. Comparison of absolute errors for *Example 1*

ε	Present method	CUM [5]	CUM [5]	FPUM [5]	FPUM [5]
	$N = 64$	$N = 64$	$N = 256$	$N = 64$	$N = 256$
$2 \cdot 10^{-10}$	$3.287 \cdot 10^{-4}$	$4.505 \cdot 10^{-3}$	$6.696 \cdot 10^{-4}$	$4.505 \cdot 10^{-3}$	$6.696 \cdot 10^{-4}$
$2 \cdot 10^{-12}$	$3.289 \cdot 10^{-6}$	$1.144 \cdot 10^{-2}$	$1.161 \cdot 10^{-3}$	$4.718 \cdot 10^{-3}$	$8.212 \cdot 10^{-4}$
$2 \cdot 10^{-14}$	$3.289 \cdot 10^{-8}$	$2.642 \cdot 10^{-2}$	$3.100 \cdot 10^{-3}$	$4.718 \cdot 10^{-3}$	$8.212 \cdot 10^{-4}$
$2 \cdot 10^{-16}$	$3.289 \cdot 10^{-10}$	$2.611 \cdot 10^{-2}$	$1.027 \cdot 10^{-2}$	$4.718 \cdot 10^{-3}$	$8.212 \cdot 10^{-4}$
$2 \cdot 10^{-18}$	$3.289 \cdot 10^{-12}$	$1.021 \cdot 10^{-2}$	$2.607 \cdot 10^{-2}$	$4.718 \cdot 10^{-3}$	$8.212 \cdot 10^{-4}$
$2 \cdot 10^{-20}$	$3.289 \cdot 10^{-14}$	$2.664 \cdot 10^{-3}$	$2.640 \cdot 10^{-2}$	$4.718 \cdot 10^{-3}$	$8.212 \cdot 10^{-4}$

Example 2. Consider the following equation [6]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t) - (2+x^2)u(x,t-1), \ (x,t) \in [0,1]\times[0,2] \\ u(x,t) = (2+x^2)\left[e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)} + e^{-\left(t+\frac{1-x}{\sqrt{\varepsilon}}\right)} \right], \ (x,t) \in [0,1]\times[-\tau,0] \\ u(0,t) = e^{-t} + e^{-t-\frac{1}{\sqrt{\varepsilon}}}, \ t\in[0,2] \\ u(1,t) = \dfrac{3}{2} e^{-t} + e^{-t-\frac{1}{\sqrt{\varepsilon}}}, \ t \in [0,2] \end{cases}$$

In this example:

$$f(x,t) = \frac{1}{2}\left[(2x\sqrt{\varepsilon} - \varepsilon) e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)} - (2x\sqrt{\varepsilon} + \varepsilon) e^{-\left(t+\frac{1-x}{\sqrt{\varepsilon}}\right)} \right]$$

the exact solution is:

$$u_T(x,t) = (2+x^2)\left[e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)} + e^{-\left(t+\frac{1-x}{\sqrt{\varepsilon}}\right)} \right]$$

the error compared with fitted operator finite difference method (FODFDM) and standard fitted difference method (SFDM) are shown in tab 2.

Table 2. Comparison of absolute errors for *Example 2*

ε	Present method	FOFDM [6]	FOFDM [6]	FOFDM [6]	SFDM [6]
	$N = 16$	$N = 16$	$N = 32$	$N = 64$	$N = 512$
10^{-8}	$6.558\cdot10^{-4}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.739\cdot10^{-3}$
10^{-10}	$6.557\cdot10^{-6}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.752\cdot10^{-5}$
10^{-12}	$6.557\cdot10^{-8}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.752\cdot10^{-7}$
10^{-14}	$6.557\cdot10^{-10}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.752\cdot10^{-9}$
10^{-16}	$6.557\cdot10^{-12}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.752\cdot10^{-11}$
10^{-18}	$6.557\cdot10^{-14}$	$1.230\cdot10^{-1}$	$6.370\cdot10^{-2}$	$3.240\cdot10^{-2}$	$2.752\cdot10^{-13}$

Conclusions and remarks

In this paper, a modified BICM is proposed for solving singularly perturbed delay partial differential equations. Numerical results compared with other methods show that the present method is simple and accurate, and it is effective for solving singularly perturbed delay partial differential equations. It is worthy to note that our method expands the application of BICM, and provides a new and efficient method for singularly perturbed delay partial differential equations. All computations are performed by the MATLABR2013A software package.

Acknowledgment

The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This paper is supported by the National Natural Science Foundation of China (No. 11361037), the Natural Science Foundation of Inner Mongolia (No. 2017MS0103, 2015MS0118), and Numerical analysis of graduate course construction project of Inner Mongolia University of Technology (KC2014001).

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Paper submitted: June 15, 2016
Paper revised: August 4, 2016
Paper accepted: September 4, 2016