A MODIFIED REPRODUCING KERNEL METHOD FOR A TIME-FRACTIONAL TELEGRAPH EQUATION

by

Yulan WANG[a], Mingjing DU[a,*], and Chaolu TEMUER[b]

[a] Department of Mathematics, Inner Mongolia University of Technology, Hohhot, China
[b] Department of Mathematics, Shanghai Maritime University, Shanghai, China

Original scientific paper
https://doi.org/10.2298/TSCI160615037W

The aim of this work is to obtain a numerical solution of a time-fractional telegraph equation by a modified reproducing kernel method. Two numerical examples are given to show that the present method overcomes the drawback of the traditional reproducing kernel method and it is an easy and effective method.

Key words: *time-fractional telegraph equation, reproducing kernel method, numerical solution*

Introduction

The mass transfer phenomenon in a boiling water reactor shows a complex hydrogen diffusion. Cazares-Ramirez and Espinosa-Paredes [1] pointed out that the mass transport with diffusion and reaction can be approximately described by the following time-fractional telegraph equation:

$$\begin{gathered}\frac{\partial^{2\alpha} c_{\mathrm{H}_2}(x,t)}{\partial t^{2\alpha}} + 2a\,\frac{\partial^{\alpha} c_{\mathrm{H}_2}(x,t)}{\partial t^{\alpha}} = D_S\,\frac{\partial^2}{\partial x^2} c_{\mathrm{H}_2}(x,t) + f(x,t),\\ 0<x<1, \qquad t>0, \qquad \frac{1}{2}<\alpha\le 1\end{gathered} \tag{1}$$

with the initial conditions:

$$c_{\mathrm{H}_2}(x,0)=0, \qquad \frac{\partial}{\partial t} c_{\mathrm{H}_2}(x,0)=0, \qquad c_{\mathrm{H}_2}(0,t)=0, \qquad \frac{\partial}{\partial x} c_{\mathrm{H}_2}(0,t)=0 \tag{2}$$

where C_{H_2} is the average concentration, a – the related with relaxation time, and D_S – the effective diffusion coefficient. The fractional derivative is defined in the Caputo sense [1]:

$$\frac{\partial^{\beta} f(t)}{\partial t^{\beta}} = \begin{cases} \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}, & \beta = n \in N,\\ \dfrac{1}{\Gamma(n-\beta)}\displaystyle\int_0^t (t-\tau)^{n-\beta-1}\,\frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, & n-1<\beta<n \end{cases} \tag{3}$$

When α = 1, eq. (1) becomes the telegraph equation, and a great error arises to describe hydrogen diffusion [1], which occurs in a scale within 1 nm, at such a small scale all

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* Corresponding author, e-mail: 724297269@qq.com

the diffusion process must be described using a fractional model. The hydrogen diffusion plays an important role in safety of nuclear power plants, and an exact description of heat and mass transfer during hydrogen generation in the core of the boiling water reactor is of great theoretical and practical importance, so it become important to solve eq. (1) effectively to elucidate its solution properties.

There are many methods to solve time-fractional differential equations, for example, the combination of a geometric approach and method of line [2], the radial basis function method [3], the reproducing kernel method [4, 5], the piecewise reproducing kernel method [6], the variational iteration method [7-10], the homotopy perturbation method [7-10], the exp-function method [7], and the fractional complex transform [11, 12]. A complete review on analytical methods for fractional calculus is available in [7]. This paper presents a modified reproducing kernel method (RKM) to solve eqs. (1) and (2). For simplification, we replace the C_{H_2} with u afterwards.

Reproducing kernel Hilbert spaces

In a space $W_2^1[0, 1] = \{u|u$ *is one-variable absolutely continuous function,* $u' \in L^2[0, 1]$, an inner product in $W_2^1[0, 1]$ is $\langle u(x), v(x)\rangle_{W_2^1} = u(0)v(0) + \int_0^1 u'(x)v'(x)\mathrm{d}x,\ u(x),\ v(x) \in W_2^1[0, 1]$, and its reproducing kernel is:

$$R_x^{\{1\}}(y) = \begin{cases} 1+x, & y > x, \\ 1+y, & x > y \end{cases} \tag{4}$$

In a space $W_2^3[0, 1] = \{u|u, u', u''$ *is one-variable absolutely continuous function,* $u(0) = u'(0) = 0, u''' \in L^2[0, 1]\}$, an inner product in $W_2^3[0, 1]$ is defined as $\langle u(x), v(x)\rangle_{W_2^3} = u''(0)v''(0) + \int_0^1 u'''(x)v'''(x)\mathrm{d}x, u(x), v(x) \in W_2^3[0, 1]$. The space $W_2^3[0, 1]$ is a reproducing kernel space and its reproducing kernel is:

$$R_x^{\{3\}}(y) = \begin{cases} \dfrac{x^2(x^3 - 5x^2y + 30y^2 + 10xy^2)}{120}, & y > x \\ \dfrac{y^2[-5xy^2 + y^3 + 10x^2(3+y)]}{120}, & x > y \end{cases} \tag{5}$$

Analytical solution and approximate solution

Let:

$$Lu(x,t) = \frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + 2a\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - D_S\frac{\partial^2}{\partial x^2}u(x,t) = f(x,t) \tag{6}$$

where
L: $W(D) \to W_1(D), D = [0,1]\times[0,1], W(D) = W_2^3[0,1]\otimes W_2^3[0,1], W_1(D) = W_2^1[0,1]\otimes W_2^1[0,1]$, L^{-1} is bounded linear operator and existent, according to [5], the exact solution is $u(x,t) = \Sigma_{j=1}^{\infty} f(x_j, t_j)\zeta_j(x,t)$ the approximate solution is $u_m(x,t) = \Sigma_{j=1}^{m} f(x_j, t_j)\zeta_j(x,t)$.

But a direct application of RKM [5] can not produce good numerical results for eqs. (1) and (2), so a modified RKM has to be adopted. Instead of continuous solution on the whole solution domain, a discrete one is searched for by dividing $x \in [0, 1]$ into M subintervals $[x_i, x_{i+1}]$, $i = 0, 1,\ldots, M-1$ with $x_0 = 0$, $x_M = 1$, $h = |x_{i+1} - x_i| = 1/M$ stretching $[x_i, x_{i+1}]$ to

[0, 1], and transforming the non-homogeneous initial conditions into homogeneous ones, so that the traditional RKM can be applied on the subintervals. It is now easy to obtain an approximate solution on $[x_i, x_{i+1}] \times [0, 1]$. Combining all the solutions of all subintervals results in an approximate solution of eqs. (1) and (2) on the entire interval. Clearly, the approximate solution is continuous on $[0, 1] \times [0, 1]$, and its error estimate can be easily done, see *e. g.* [6].

Numerical verification

Example 1. Consider eqs. (1) and (2) with $\alpha = 0.6$, $a = 0.5$, $D_S = 1$, and the exact solution is $u(x, t) = t^3 \sin x$. Pinking $x_i = i/n$, $t_i = i/n$ on $D = [0, 1] \times [0, 1]$, $n = 10$, by MATHEMATICA 7.0, the exact solution is shown in fig. 1. The absolute errors by traditional RKM [5] (1) and the present method $h = 0.1$, (2) for $t = 1$ are shown in fig. 2. It is obviously shown that the present method is more accurate than traditional RKM. Figure 3 shows the absolute errors by the present method when $h = 0.1$ (1), 0.01 (2), and 0.001 (3) for $t = 1$. It is obviously shown that the more pieces are chosen, the more accuracy can be achieved. Figure 4 shows the absolute errors by the present method when $h = 0.1$ (1), 0.01 (2), and 0.001 (3) for x = 0.1. Table 1 shows that the numerical comparison of absolute errors of by two methods.

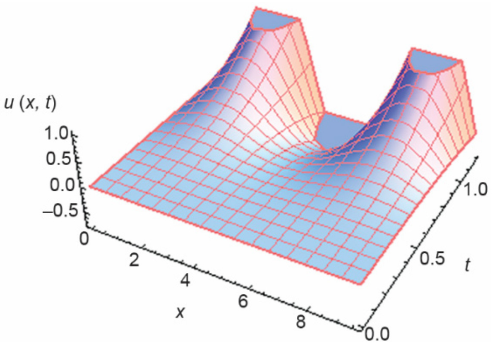

Figure 1. Exact solution of *Example 1*

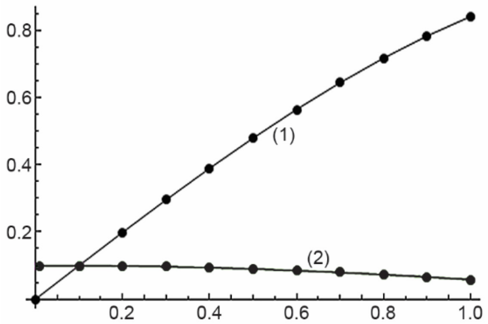

Figure 2. Absolute errors of *Example 1* by traditional RKM (1) and present method, $h = 0.1$ (2) for $t = 1$

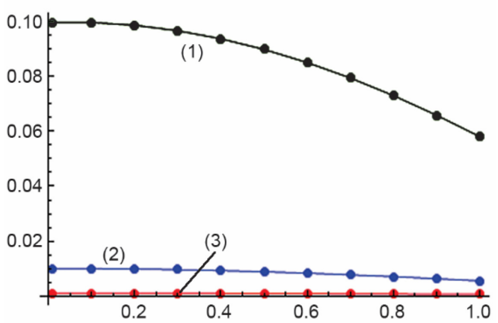

Figure 3. Absolute errors of *Example 1* by present method when $h = 0.1$ (1), $h = 0.01$ (2), and $h = 0.001$ (3) for $t = 1$

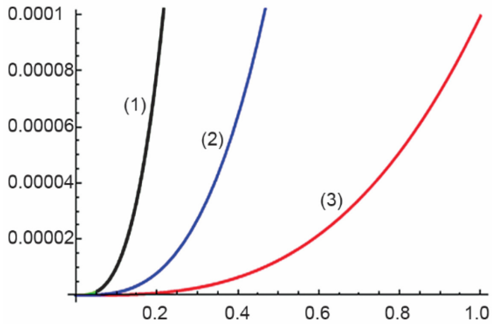

Figure 4. Absolute errors of *Example 1* by present method when $h = 0.1$ (1), $h = 0.01$ (2), and $h = 0.001$ (3) for $x = 0.1$

Table 1. Numerical comparison of absolute errors of *Example 1* by two methods

$u(x_i, t_i)$	Traditional RKM [5]	Present method $h = 0.1$	Present method $h = 0.01$	Present method $h = 0.001$
$u(x_1, t_1)$	9.98334E-05	9.98334E-05	9.95487E-06	9.95054E-07
$u(x_2, t_2)$	1.58935E-03	7.90687E-04	7.84835E-05	7.84133E-06
$u(x_3, t_3)$	7.97905E-03	2.61497E-03	2.58336E-04	2.57981E-05
$u(x_4, t_4)$	2.49228E-02	6.00948E-03	5.90715E-04	5.89604E-05
$u(x_5, t_5)$	5.99282E-02	1.12509E-02	1.09996E-03	1.09728E-05
$u(x_6, t_6)$	1.21963E-01	1.84069E-02	1.78879E-03	1.78333E-04
$u(x_7, t_7)$	2.20967E-01	2.72943E-02	2.63441E-03	2.62451E-04
$u(x_8, t_8)$	3.67286E-01	3.74469E-02	3.58544E-03	3.56897E-04
$u(x_9, t_9)$	5.71045E-01	4.80927E-02	4.56001E-03	4.53439E-04
$u(x_{10}, t_{10})$	8.41471E-01	5.81441E-02	5.44501E-03	5.40723E-04

Example 2. Consider eqs. (1) and (2) with $\alpha = 0.8$, $a = 0.5$, $D_S = 1$ the exact solution is $u(x, t) = \sin t^2 \sin^2 x$. The numerical results are show in figs. 5-8. Table 2 shows that the numerical comparison of absolute errors of by two methods.

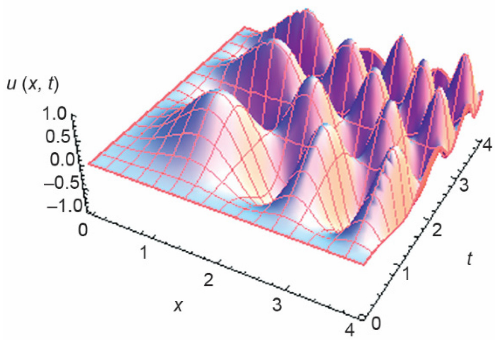

Figure 5. Exact solution of *Example 2*

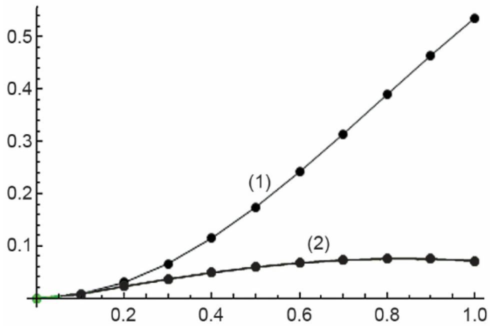

Figure 6. Absolute errors of *Example 2* by traditional RKM (1) and present method, $h = 0.1$ (2) for $t = 2$

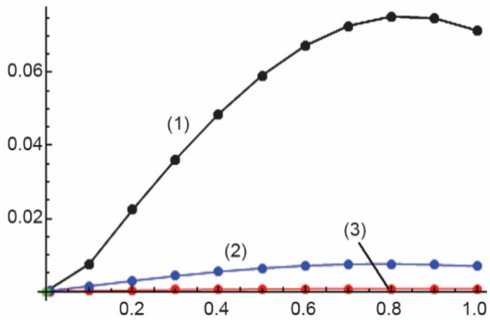

Figure 7. Absolute errors of *Example 2* by the present method when $h = 0.1$ (1), $h = 0.01$ (2), and $h = 0.001$ (3) for $t = 2$

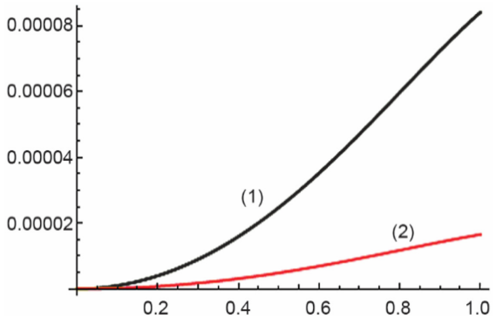

Figure 8. Absolute errors of *Example 2* by the present method when $h = 0.1$ (1), and $h = 0.001$ (2) for $x = 0.1$

Table 2. Numerical comparison of absolute errors of *Example 2*

$u(x_i, t_i)$	Traditional RKM [5]	Present method $h = 0.1$	Present method $h = 0.01$	Present method $h = 0.001$
$u(x_1, t_1)$	9.96654E-05	9.96654E-05	1.88853E-05	1.97686E-06
$u(x_2, t_2)$	1.57836E-03	1.17980E-03	1.52032E-04	1.55357E-05
$u(x_3, t_3)$	7.84929E-03	4.30183E-03	5.00041E-04	5.06750E-05
$u(x_4, t_4)$	2.41601E-02	1.02465E-02	1.13170E-03	1.14177E-04
$u(x_5, t_5)$	5.68655E-02	1.93475E-02	2.06822E-03	2.08049E-04
$u(x_6, t_6)$	1.12312E-01	3.13426E-02	3.27035E-03	3.28205E-04
$u(x_7, t_7)$	1.95317E-01	4.52720E-02	4.62947E-03	4.63698E-04
$u(x_8, t_8)$	3.07317E-01	5.94707E-02	5.97075E-03	5.96958E-04
$u(x_9, t_9)$	4.44423E-01	7.17054E-02	7.06944E-03	7.05509E-04
$u(x_{10}, t_{10})$	5.95823E-01	7.94958E-02	7.68598E-03	7.22655E-04

Conclusion

The hydrogen diffusion in nuclear power plants can be effectively controlled by the fractional order involved in eq. (1). In the hydrogen scale, the diffusion medium must be considered as a fractal medium, and the fractional order is directly relevant to the value of the fractal dimensions [13]. In this paper, a modified RKM is employed successfully for solving a class of time-fractional telegraph equations which can describe hydrogen diffusion and reaction in boiling water reactor. The numerical results of two examples show that the modified method is more accurate and reliable than the traditional RKM. Moreover, the modified RKM is also effective for solving other partial differential equations.

Acknowledgment

The authors would like to express their thankfulness to the unknown referees for their careful reading and helpful comments. This paper is supported by the National Natural Science Foundation of China (No. 11361037), Natural Science Foundation of Inner Mongolia (No. 2017MS0103, No. 2015MS0118), and Numerical analysis of graduate course construction project of Inner Mongolia University of Technology (No. KC2014001).

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Paper submitted: June 15, 2016
Paper revised: August 4, 2016
Paper accepted: October 25, 2016