COMPACT SCHEMES FOR KORTEWEG-DE VRIES EQUATION

by

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This paper proposes one family of compact schemes for Korteweg-de Vries equation. In the deterministic case, the schemes are convergent with fourth-order accuracy both in space and in time. Moreover, the schemes are stable. The numerical dispersion relation is analyzed. We compare the schemes with one second-order scheme. The numerical examples test the effect of the schemes. In the stochastic case, we simulate the wave profile and three discrete dynamical quantities for Korteweg-de Vries equation with small noise. The white noise has stochastic influence on the profile and dynamical quantities of the solution. If the size of noise increases, the perturbation on the profile and dynamical quantities will increase accordingly.

Key words: *Korteweg-de Vries equation, compact scheme, stability*

Introduction

Korteweg-de Vries (KdV) equation and its modifications are widely used to study shallow water wave with small amplitude or with weakly non-linear restoring force, MHD wave in collision less plasma, bubble-liquid mixing wave, and non-linear long wave in non-harmonic lattice [1]. The KdV equations describe the interaction and balance of dispersion term and non-linear term. There exist stable solitons and reconstruction of initial waveform for KdV equations. Some theoretical analysis and numerical simulation for KdV equations have been proposed, for examples, the variational iteration method [2-4], the homotopy perturbation method [2, 5], the exp-function method [6, 7], and first integral method [8], symplectic scheme [9, 10], finite element method [11], the meshless method of lines [12], and finite difference method [13].

Recently compact schemes, which possess high accuracy, compactness, and economic resource, are widely used in scientific computation [14-18]. A compact splitting multi-symplectic scheme for some Schrodinger equations is proposed in [15]. Kanazawa *et al.* [17] investigates a conservative compact finite difference scheme for the deterministic KdV equation. A pseudo-compact scheme for the Rosenau-KdV equation coupling with the Rosenau-RLW equation is analyzed in [18].

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It is found that, for stochastic KdV equations, the reference of compact schemes is lacking. The compact scheme is valid for deterministic KdV equation and its theory has already been well established [17, 18], but what will happen for its stochastic partner? How will the compact scheme behave? The two questions motivate us to design efficient schemes for stochastic KdV equation. By applying compact operators, we construct one-parameter family of compact schemes to an initial value problem of the following stochastic KdV equation:

$$\begin{cases} u_t + uu_x + \varepsilon u_{xxx} = \mu\dot{\chi}, & x \in [0, L] \\ u(x,0) = g(x), & t \in [0, T] \end{cases} \tag{1}$$

where $g(x)$ is a differential function, u_t, u_x, and u_{xxx} are the mean partial derivatives of u with respect to t and x, respectively, μ is a small real number, and $\dot{\chi}$ – a real-valued white noise which is delta correlated in time, either smooth or delta correlated in space. For KdV equations, uu_x, εu_{xxx}, and $\mu\dot{\chi}$ are non-linear term, dispersion term, and stochastic disturbance term, respectively. If $\mu = 0$, the system (1) is a deterministic system and preserves some physical quantities [19] under periodic condition:

$$H_1(t) = \int_0^L u(x,t)^3 - 3\varepsilon u_x(x,t)^3 \,\mathrm{d}x, \qquad H_2(t) = \int_0^L u(x,t)^2 \,\mathrm{d}x, \qquad H_3(t) = \int_0^L u(x,t)\mathrm{d}x \tag{2}$$

These physical quantities can be used to test the efficiency of numerical methods for the stochastic KdV equation.

Compact schemes

Further in the text, by applying a kind of compact operators, we present a kind of compact schemes for KdV equation (1). For simplicity, we consider the uniform mesh grids $\{x_k = kh, t_n = n\tau\}$ with step-sizes $h = L/N$ and $\tau = T/M$. Numerical values of $u(x_k, t_n)$ are denoted by u_n^k, u_k, and u^n stand for solution vectors at $x = x_k$ and $t = t^n$ with components $u(x_k, t)$ and $u(x, t_n)$, respectively. For spatial discretization of eq. (1) under periodic condition, we consider the linear operators $Au_k = \alpha u_{k-1} + u_k + \alpha u_{k+1}$ and:

$$Bu_k = b\frac{u_{k+2} - u_{k-2}}{4h} + a\frac{u_{k+1} - u_{k-1}}{2h}, \qquad Cu_k = a\frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-2}}{2h^3} \tag{3}$$

We adopt the operators:

$$\delta_x u_k = A^{-1}Bu_k, \delta_{3x}u_k = A^{-1}Cu_k \tag{4}$$

to approximate u_x and u_{xxx}, respectively [14, 15]. Denote four cyclic matrices by:

$$M_1 = \begin{pmatrix} 1 & \alpha & & & \alpha \\ \alpha & 1 & \alpha & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & 1 & \alpha \\ \alpha & & & \alpha & 1 \end{pmatrix}_{N\times N}, M_0(k_1,k_2) = \begin{pmatrix} 0 & k_1 & k_2 & & & -k_2 & -k_1 \\ -k_1 & 0 & k_1 & k_2 & & & -k_2 \\ -k_2 & -k_1 & 0 & k_1 & k_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -k_2 & -k_1 & 0 & k_1 & k_2 \\ k_2 & & & -k_2 & -k_1 & 0 & k_1 \\ k_1 & k_2 & & & -k_2 & -k_1 & 0 \end{pmatrix}_{N\times N} \tag{5}$$

$M_2 = M_0(a/2h, b/4h), M_3 = (a/2h^3)M_0(-2,1)$. Then, under periodic conditions, the matrix form of (4) is:

$$\delta_x u_k = M_1^{-1} M_2 u_k, \qquad \delta_{3x} u_k = M_1^{-1} M_3 u_k \tag{6}$$

One-parameter family of fourth-order approximations δ_x to u_x are defined by $3a = 4 + 2\alpha$, $3b = 4\alpha - 1$. When $a = 2$, $\alpha = 0.5$, a fourth order approximation δ_{3x} to u_{xxx} is predicted. Taylor analysis yields that, the leading truncation errors of δ_x and δ_{3x} are $4/5!(3\alpha - 1)(u_{xxxxx})_k h^4$ and $42/7!(u_{xxxxxxx})_k h^4$, respectively. The Fourier analysis yields that:

$$\delta_x u_k = iv_1 u_k, \qquad \delta_{3x} u_k = iv_2 u_k \tag{7}$$

where

$$v_1 = \frac{2a\sin(\beta h) + b\sin(2\beta h)}{2h[1 + 2\alpha\cos(\beta h)]}, \qquad v_2 = \frac{2a\sin(\beta h)[\cos(\beta h) - 1]}{h^3[1 + 2\alpha\cos(\beta h)]} \tag{8}$$

Discretizing KdV equations (1) in spatial direction with compact operators (4) yields the following ordinary differential equation of $z(t) = u(x_k, t)$:

$$z' + z\delta_x z + \varepsilon\delta_{3x} z = \mu\dot{\chi}_k \tag{9}$$

By applying famous 4-stage Runge-Kutta method [20] to eq. (9) we can get:

$$\begin{aligned} z^{n+1} &= z^n + \frac{\tau}{2}[f(T_1, Y_1) + f(T_2, Y_2)] \\ Y_1 &= z^n + \tau\left[\frac{1}{4} f(T_1, Y_1) + \left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right) f(T_2, Y_2)\right] \\ Y_2 &= z^n + \tau\left[\left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right) f(T_1, Y_1) + \frac{1}{4} f(T_2, Y_2)\right] \end{aligned} \tag{10}$$

where $f(t,z) = -z\delta_x z - \varepsilon\delta_{3x} z + \mu\dot{\chi}_k$, $T_1 = t_n + (3 - \sqrt{3})\tau/6$, $T_2 = t_n + (3 + \sqrt{3})\tau/6$.

Deterministic analysis

Consider the deterministic case that $\mu = 0$. Suppose that $u_k^n = u(x_k, t_n)$. Taylor analysis yields that the compact schemes (10) are convergent with 4-order accuracy both in time and in space. The schemes are most compact tridiagonal schemes with minimal spatial grid. The leading terms of local truncation errors of (10) are:

$$\frac{4}{5}!(3\alpha - 1)(u_{xxxxx})_k^n h^4 + \frac{42}{7}!(u_{xxxxxxx})_k^n h^4 +$$

$$+ \tau^4 \left[\frac{-f^4 f^{(4)}}{2160} - 31\frac{f^3 f^{(3)} f'}{540} + 13\frac{f^3 f''^2}{1440} + (5\sqrt{3} - 333)\frac{f^2 f'^2 f''}{1440} - 17\frac{ff'^4}{360}\right]_k^n$$

Consider the compact schemes (10) applied to the linear case $u_t + pu_x + qu_{xxx} = 0$. By computing we get that:

$$u^{n+1} = \frac{\tau^2(p\delta_x + q\delta_{3x})^2 - 6\tau(p\delta_x + q\delta_{3x}) + 12}{\tau^2(p\delta_x + q\delta_{3x})^2 + 6\tau(p\delta_x + q\delta_{3x}) + 12} u^n$$

Since the eigenvalues of compact operators δ_x, δ_{3x} are zeros or imaginary numbers, the modules of multiplying matrices are 1 and so the schemes (10) are linearly stable.

Then we consider the dispersion relation in linear case. Suppose that the exact solution is in the form:

$$u(x,t) = \exp\left[i\left(\frac{\xi}{h}x + \frac{\omega}{\tau}t \right) \right]$$

where ξ is the wave number and ω – the frequency. Then by inserting it into the linear equation we can get the exact dispersion relation:

$$\omega + c\xi - \frac{c}{d}\xi^3 = 0 \tag{11}$$

where $c = p\tau/h$ and $d = ph^2/q$.

Assume that the formal numerical solution is $u_k^n = \exp[i(k\xi + n\omega)]$.

By computing according to eq. (10), we get the following numerical dispersion relation:

$$\sin\omega = \frac{12A(A^2 - 12)}{A^4 + 12A^2 + 144} \tag{12}$$

where

$$A = \frac{c}{d}\frac{2(\sin 2\xi - 2\sin\xi)}{1+\cos\xi} + c\frac{3\sin\xi}{2+\cos\xi}$$

Second-order scheme

If $b = 0, a = 1,$ and $\alpha = 0,$ then the compact operators δ_x and δ_{3x} are second-order with the leading truncation errors $(1/6)(u_{xxx})_k h^2$ and $(1/4)(u_{xxxxx})_k h^2$, respectively. By applying the forward difference formula of first derivative and the averaging operator to eq. (9), we can get that:

$$\frac{u_k^{n+1} - u_k^n}{\tau} + (u\delta_x u)_k^{n+1/2} + \varepsilon\delta_{3x}u_k^{n+1/2} = \mu\dot{\chi}_k^{n+1/2} \tag{13}$$

where

$$u_k^{n+1/2} = \frac{1}{2}(u_k^{n+1} + u_k^n), \qquad \dot{\chi}_k^{n+1/2} = \frac{1}{\sqrt{h\tau}}\chi_k^{n+1/2}$$

where $\chi_k^{n+1/2}$ is a sequence of independent random variables with normal distribution law N(0, 1) [21]. This means that we discretize eq. (9) in temporal direction:

$$u_k^{n+1} - u_k^n + \tau(u\delta_x u)_k^{n+1/2} + \tau\varepsilon\delta_{3x}u_k^{n+1/2} = \tau\mu\dot{\chi}_k^{n+1/2} \tag{14}$$

Consider the deterministic case that $\mu = 0$. Suppose that $u_k^n = u(x_k, t_n)$. Taylor analysis yields that the scheme (13) is convergent with second-order accuracy both in time and in space. The leading terms of local truncation errors of eq. (13) is:

$$\frac{1}{6}(u_{xxx})_k^n h^2 + \frac{1}{4}(u_{xxxxx})_k^n h^2 + \frac{\tau^2}{24}(f''f^2 - 2f'^2 f)_k^n$$

The scheme (14) applied to linear equation generates the formula:

$$2(u^{n+1}-u^n)+p\tau\delta_x(u^n+u^{n+1})+q\tau\delta_{3x}(u^n+u^{n+1})=0$$

which means that:

$$4u_k^{n+1}+c\left(u_{k+1}^{n+1}-u_{k-1}^{n+1}\right)+\frac{c}{d}\left(u_{k+2}^{n+1}-2u_{k+1}^{n+1}+2u_{k-1}^{n+1}-u_{k-2}^{n+1}\right)=$$
$$=4u_k^{n}-c\left(u_{k+1}^{n}-u_{k-1}^{n}\right)-\frac{c}{d}\left(u_{k+2}^{n}-2u_{k+1}^{n}+2u_{k-1}^{n}-u_{k-2}^{n}\right)$$

Therefore, the module of multiplying matrix is 1 and so the scheme (14) is linearly stable.

Then we consider the dispersion relation in linear case. By computing according to eq. (14), we get numerical dispersion relation:

$$\sin\omega=\frac{-4B}{B^2+4} \tag{15}$$

where $B=c\sin\xi+(c/d)(\sin 2\xi-2\sin\xi)$.

Non-periodic case

Now we consider spatial discretization under non-periodic condition. At some discrete points near to the boundary, we apply the following compact operators for scheme (10):

$$\delta_x u_1+3\delta_x u_2=\frac{-17u_1+9u_2+9u_3-u_4}{6h},\quad \delta_x u_N+3\delta_x u_{N-1}=\frac{17u_N-9u_{N-1}-9u_{N-2}+u_{N-3}}{6h},$$
$$\delta_{3x}u_1-15\delta_{3x}u_2=\frac{1}{h^3}(22u_1-76u_2+98u_3-58u_4+16u_5-2u_6),$$
$$\delta_{3x}u_N-15\delta_{3x}u_{N-1}=\frac{1}{h^3}(-22u_N+76u_{N-1}-98u_{N-2}+58u_{N-3}-16u_{N-4}+2u_{N-5}),$$
$$-\frac{1}{14}\delta_{3x}u_1+\delta_{3x}u_2-\frac{1}{14}\delta_{3x}u_3=\frac{1}{7h^3}(-10u_1+35u_2-46u_3+28u_4-8u_5+u_6),$$
$$-\frac{1}{14}\delta_{3x}u_N+\delta_{3x}u_{N-1}-\frac{1}{14}\delta_{3x}u_{N-2}=\frac{1}{7h^3}(10u_N-35u_{N-1}+46u_{N-2}-28u_{N-3}+8u_{N-4}-u_{N-5})$$

Meanwhile, for scheme (14), we give the discrete spatial operator at boundary points:

$$\delta_x u_1=\frac{-3u_1+4u_2-u_3}{2h},\quad \delta_x u_N=\frac{3u_N-4u_{N-1}+u_{N-2}}{2h},$$
$$\delta_{3x}u_1=\frac{1}{2h^3}(-5u_1+18u_2-24u_3+14u_4-3u_5),$$
$$\delta_{3x}u_N=\frac{1}{2h^3}(5u_N-18u_{N-1}+24u_{N-2}-14u_{N-3}+3u_{N-4}),$$
$$\delta_{3x}u_2=\frac{1}{2h^3}(-3u_1+10u_2-12u_3+6u_4-u_5),$$
$$\delta_{3x}u_{N-1}=\frac{1}{2h^3}(3u_N-10u_{N-1}+12u_{N-2}-6u_{N-3}+u_{N-4})$$

Numerical examples

Now we apply the schemes (10) and (14) to solve the KdV equation (1) with:

$$\varepsilon = 4.84e-4, \qquad L = 2, \qquad K_1 = 0.3, \qquad K_2 = \frac{\sqrt{\dfrac{K_1}{\varepsilon}}}{2}, \qquad g = 3K_1 \operatorname{sec}h^2(K_2 x - 6)$$

For the sake of simplicity, we choose $\alpha = 1/4$ for δ_x to eq. (12). Numerical results are similar to other different α.

First to test the efficiency of the schemes in the deterministic case, we suppose that $\mu = 0$. In fact, in this deterministic case, the exact solution of the system (1) is:

$$u = 3K_1 \operatorname{sec}h^2(K_2 x - K_1 K_2 t - 6)$$

In fig. 1 we depict the exact dispersion relation (11) and numerical dispersion relations (12) and (15) corresponding to $c = 0.2$ and different d. The numerical dispersion curve of scheme (10), relative to scheme (14), is closer to the exact curve.

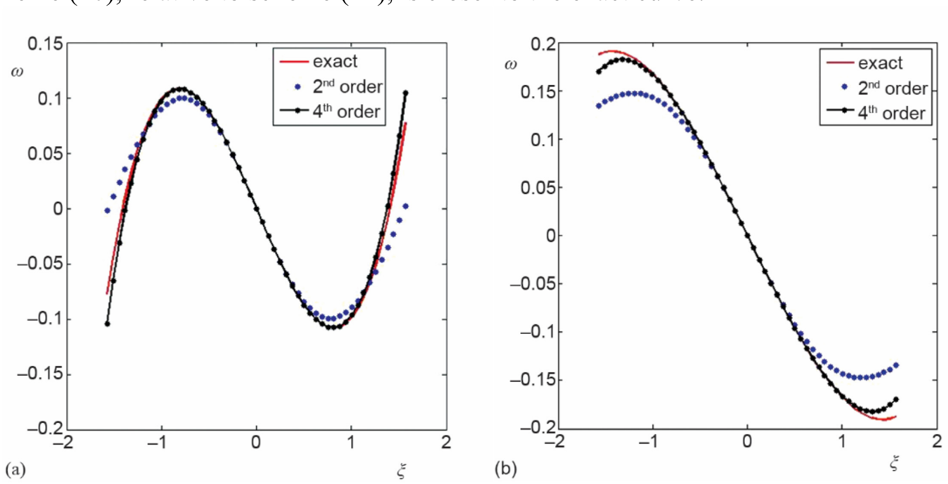

Figure 1. Dispersion relation curves d = 1.98 (a), d = 6.13 (b)

In fig. 2 we plot the second-norm errors between numerical solutions and exact solution at $t = 1$ with $h = 0.05$, $\tau = 0.001$. The errors conform that the schemes (10) are more accurate than the scheme (14).

We define three discrete dynamical quantities for $H_1(t)$, $H_2(t)$, and $H_3(t)$:

$$H_1^n = h\sum\nolimits_k [(u_k^n)^3 - 3\varepsilon(\delta_x u_k^n)^3], \qquad H_2^n = h\sum\nolimits_k (u_k^n)^2, \qquad H_3^n = h\sum\nolimits_k u_k^n \tag{16}$$

Denote the residual errors of H_1^n, H_2^n and H_3^n by:

$$\Delta H_1^n = H_1^n - H_1(t_0), \qquad \Delta H_2^n = H_2^n - H_2(t_0), \qquad \Delta H_3^n = H_3^n - H_3(t_0) \tag{17}$$

respectively. From figs. 3-5, we depict the residual errors $\Delta H_1^n, \Delta H_2^n$, and ΔH_3^n for numerical solution with $h = 0.05$, $\tau = 0.001$. These figures show that for deterministic KdV equation, the compact schemes (10) simulate the discrete dynamical quantities (16) more approximately than the scheme (14).

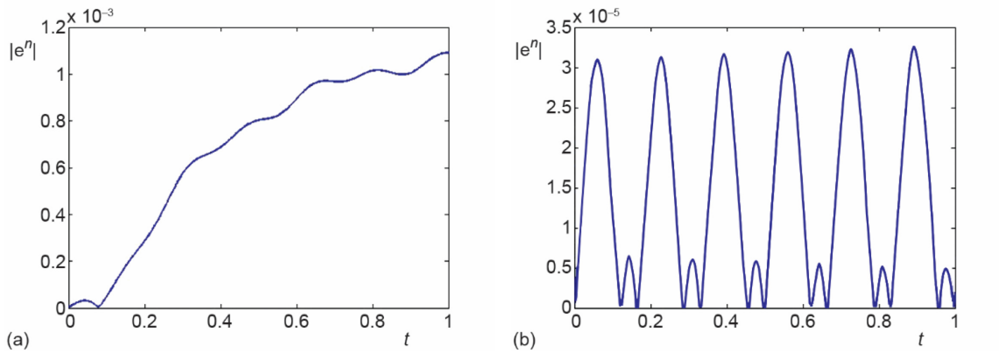

Figure 2. Errors of numerical solutions for scheme (10) (b) and scheme (14) (a)

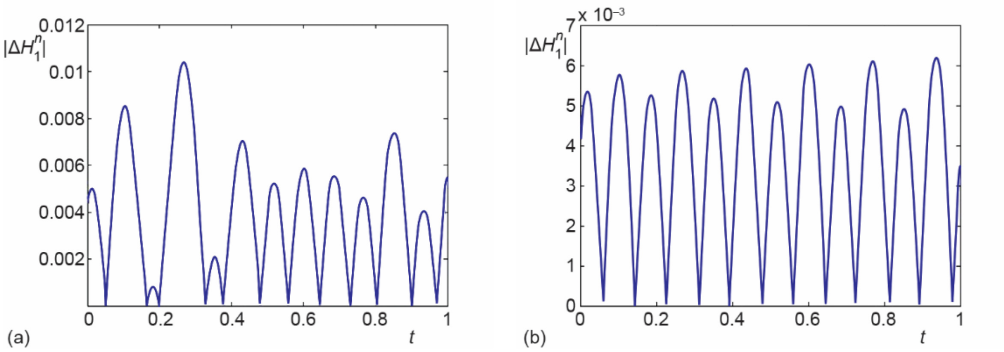

Figure 3. Errors of H_1 for scheme (10) (b) and scheme (14) (a)

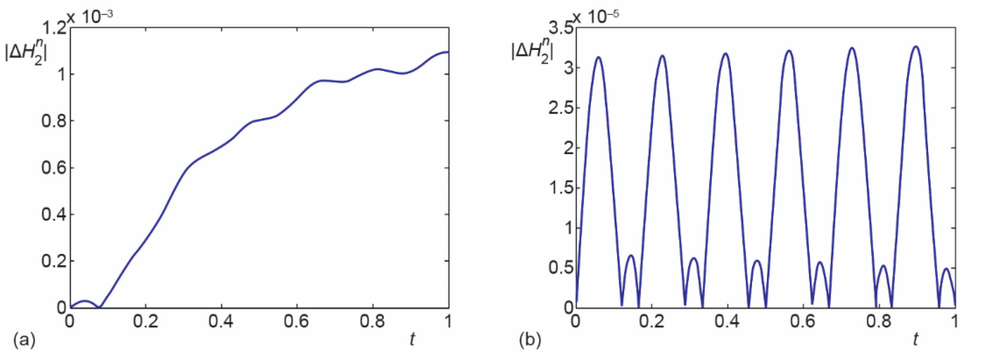

Figure 4. Errors of H_2 for scheme (10) (b) and scheme (14) (a)

Next we simulate the profile of stochastic KdV wave with small μ. In fig. 6 we plot the 3-D profile of $|u_k^n|$ with scheme (10) for one trajectory with h = 0.05, τ = 0.0001, μ = 0.025 (b), and μ = 0.009 (a). From $|u_k^n|$ depicted in the figure, we find that the white noise produces stochastic influence on the KdV waves.

At last we simulate the stochastic influence of white noise on the physical quantities. From figs. 7-9, we depict the discrete quantities (16) for numerical solution with (10) of one trajectory with $h = 0.05$, $\tau = 0.0001$, $\mu = 0.05$ (b), and $\mu = 0.009$ (a).

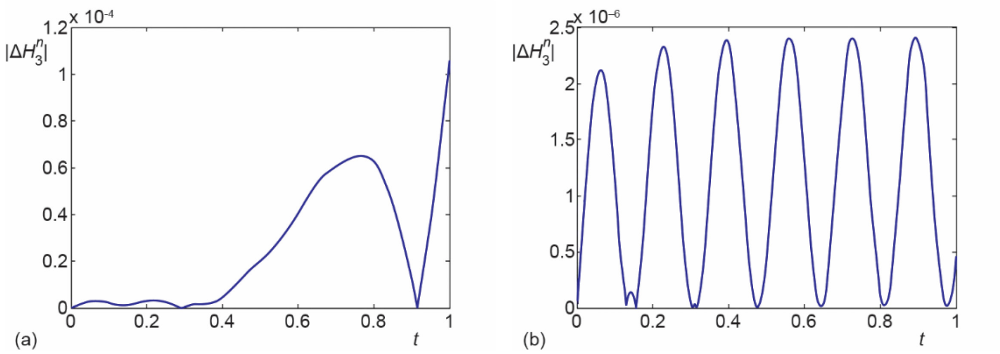

Figure 5. Errors of H_3 for scheme (10) (b) and scheme (14) (a)

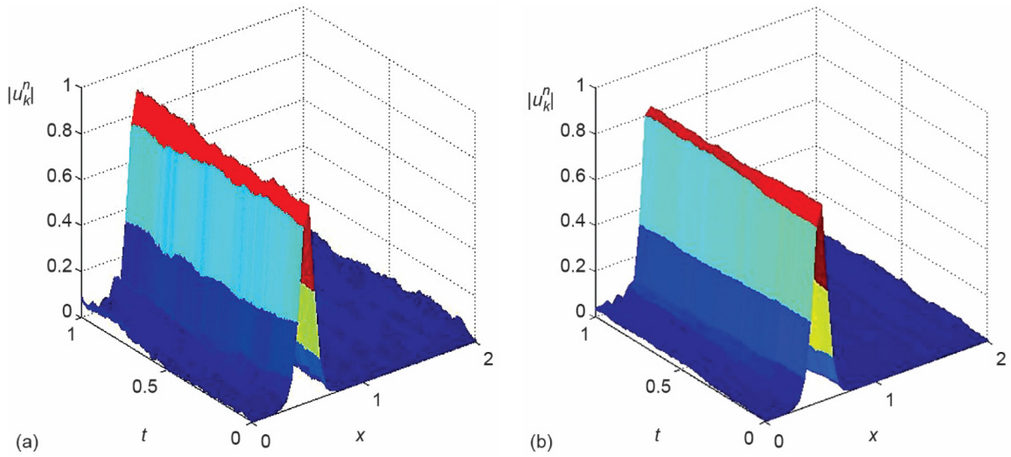

Figure 6. The 3-D profile for μ = 0.025 (b) and μ = 0.009 (a)

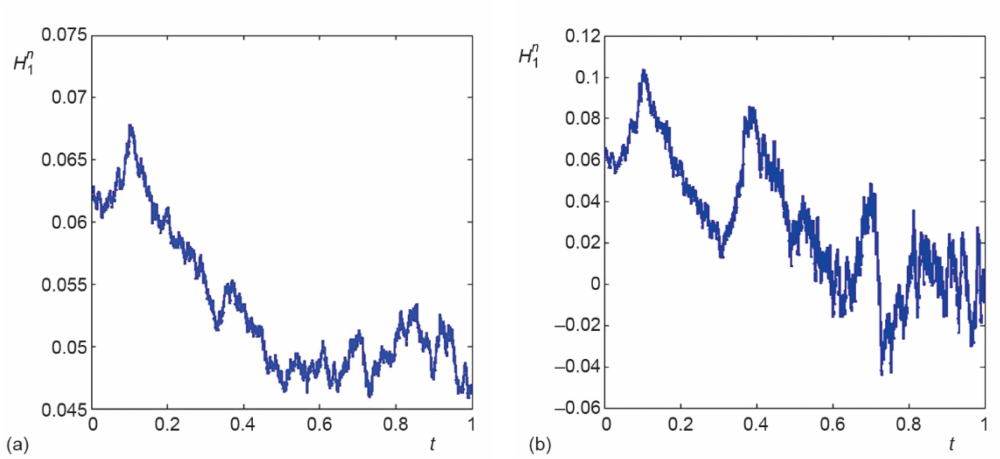

Figure 7. The H_1^n for numerical solution with μ = 0.05 (b) and μ = 0.009 (a)

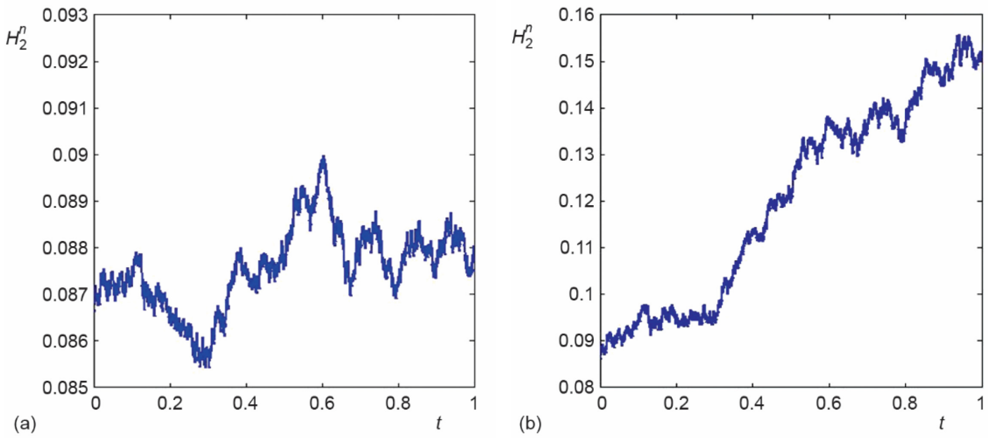

Figure 8. The H_2^n for numerical solution with μ = 0.05 (b) and μ = 0.009 (a)

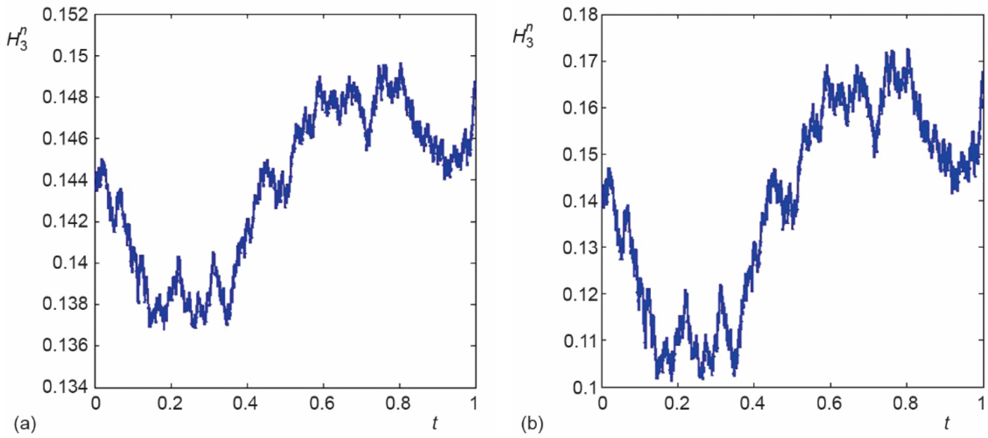

Figure 9. The H_3^n for numerical solution with μ = 0.05 (b) and μ = 0.009 (a)

From the results depicted in figs. 7-9, we can find that the white noise produces stochastic perturbation on the three discrete dynamical quantities. If the size of noise increases, the perturbation on the profile and dynamical quantities will increase accordingly. The larger the size of noise is, the more the perturbation will be.

Conclusion

For KdV equation, we present compact schemes (10) with minimal spatial grids. They are stable and convergent with fourth-order accuracy in each direction for deterministic KdV equation. Compared with scheme (14), the schemes (10) are more efficient (see figs. 1-5) in simulating the solutions, dispersion relation and physical quantities. For stochastic equation, the white noise has stochastic perturbation upon the wave profile and discrete dynamical quantities (see figs. 6-9). We observe that the perturbation will increase with the size of noise. Next, further discussion and research about the strong and weak convergence for the schemes will be needed.

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