EXACT TRAVELING WAVE SOLUTIONS FOR A NEW NON-LINEAR HEAT TRANSFER EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI160512076G

In this paper, we propose a new non-linear partial differential equation to describe the heat transfer problems at the extreme excess temperatures. Its exact traveling wave solutions are obtained by using Cornejo-Perez and Rosu method.

Key words: *traveling wave solution, non-linear heat transfer equation, Cornejo-Perez and Rosu method, exact solution, heat transfer*

Introduction

The approximate, analytical, numerical, and exact solutions for the non-linear heat transfer problems have been investigated by a great many of engineers and scientists for many years [1-2]. Many technologies were proposed to find the solutions for them, such as the homotopy perturbation method [3], heat-balance integral method [4], variational iteration method [5], and integral transforms [6], and so on.

Recently, Cornejo-Perez and Rosu [7] proposed a new method to find the exact solutions to non-linear ordinary differential equations (ODE) and to apply to derive the exact traveling wave solutions for the non-linear partial differential equations (PDE) [8]. The targets of the previous paper are to address a new non-linear heat transfer equation and to find its exact traveling wave solutions.

Mathematical model proposed

Let us consider the non-linear heat transfer equation [1, 4]:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} - \alpha \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2} = \Lambda[\Theta(\xi,\tau)] \tag{1}$$

where α [$\text{Wm}^{-1}\text{K}^{-1}$] is the heat diffusion coefficient and $\Lambda[\Theta(\xi, \tau)]$ – the non-linear heat source.

When $\Lambda[\Theta(\xi, \tau)]$ is expanded for the constants C_i determined by the temperature field:

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$$\Lambda[\Theta(\xi,\tau)] = \sum_{i=0}^{n} C_i \Theta^i(\xi,\tau) \tag{2}$$

Equation (1) can be re-written:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} - \alpha \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2} = \sum_{i=0}^{n} C_i \Theta^i(\xi,\tau) \tag{3}$$

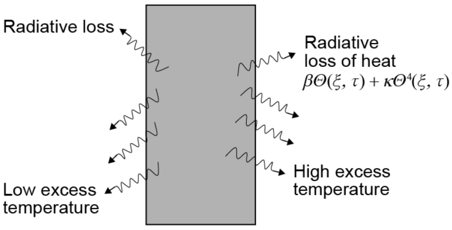

Figure 1. The extreme excess temperature field

As a special case of eq. (2), we consider the radiative loss of heat at the extreme excess temperature field shown in fig. 1, denoted by $\Lambda(\xi, \tau)$:

$$\Lambda(\xi,\tau) = \beta\Theta(\xi,\tau) + \kappa\Theta^4(\xi,\tau) \tag{4}$$

where β and κ are constants, and $\Theta(\xi, \tau)$ is the excess temperature field.

The governing equation at the extreme excess temperature can be written as:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2} - \beta\Theta(\xi,\tau) - \kappa\Theta^4(\xi,\tau), \qquad (\xi,\tau) \in [0,\infty) \times [0,\infty) \tag{5}$$

At the low excess temperature, eq. (2) reads [9]:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2} - \beta\Theta(\xi,\tau), \qquad (\xi,\tau) \in [0,\infty) \times [0,\infty) \tag{6}$$

where β is a constant and $\Theta(\xi, \tau)$ – the low excess temperature field.

At the high excess temperature, eq. (2) can be re-written [10]:

$$\frac{\partial \Theta(\xi,\tau)}{\partial \tau} = \alpha \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2} - \kappa\Theta^4(\xi,\tau), \qquad (\xi,\tau) \in [0,\infty) \times [0,\infty) \tag{7}$$

where κ is related to Stefan's constant and $\Theta(\xi, \tau)$ – the excess temperature field.

Equation (1) adopting in the previous paper is called the heat transfer equation at the extreme excess temperature and the parameters β and κ are determined by the different temperature fields.

Analysis of method

In this section, we introduce the traveling wave transformation method based on theory of Cornejo-Perez and Rosu group [7, 8].

In order to illustrate the methodology, we consider the PDE with respect to ξ and τ given by:

$$\aleph\left[\frac{\partial \Theta(\xi,\tau)}{\partial \tau}, \frac{\partial^2 \Theta(\xi,\tau)}{\partial \xi^2}, \frac{\partial \Theta(\xi,\tau)}{\partial \xi}, \Theta(\xi,\tau), \Theta^k(\xi,\tau)\right] = 0 \tag{8}$$

where k is a positive integer.

The TTM can be written:

$$\theta = \xi - \vartheta\tau \tag{9}$$

where ϑ is a constant.

Due to the chain rules:

$$\frac{\partial\Theta(\xi,\tau)}{\partial\tau} = -\vartheta\frac{\partial\Theta(\xi,\tau)}{\partial\theta} \quad \text{and} \quad \frac{\partial^2\Theta(\xi,\tau)}{\partial\xi^2} = \frac{\partial^2\Theta(\theta)}{\partial\theta^2} \tag{10a,b}$$

Equation (8) becomes the ODE with respect to θ given by:

$$\aleph\left[\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\xi^2}, \frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\xi}, \Theta(\theta), \Theta^k(\theta)\right] = 0 \tag{11}$$

Following the Cornejo-Perez and Rosu group [7, 8], eq. (11), can be written:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta} - v_1(\Theta)\right]\left[\frac{\mathrm{d}}{\mathrm{d}\theta} - v_2(\Theta)\right]\Theta = 0 \tag{12}$$

which reduces to the following equation of the factorized form

$$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} - \left[\frac{\mathrm{d}v_1(\Theta)}{\mathrm{d}\Theta}\Theta + v_1(\Theta) + v_2(\Theta)\right]\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} + v_1(\Theta)v_2(\Theta)\Xi(\Theta) = 0 \tag{13}$$

which leads to:

$$\frac{\mathrm{d}v_1(\Theta)}{\mathrm{d}\Theta}\Theta + v_1(\Theta) + v_2(\Theta) = \Lambda(\Theta), \qquad v_1(\Theta)v_2(\Theta) = \frac{\Pi(\Theta)}{\Theta} \tag{14a,b}$$

where $\Theta(\theta) = \Theta$, $v_1[\Theta(\vartheta)] = v_1(\Theta)$, $v_2[\Theta(\vartheta)] = v_2(\Theta)$, $\Lambda[\Theta(\vartheta)] = \Lambda(\Theta)$, and $\Pi[\Theta(\vartheta)] = \Pi(\Theta)$.

Thus, eq. (12) can be re-written as the second order ODE with respect to θ given by:

$$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} - \Lambda(\Theta)\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} + \Pi(\Theta) = 0 \tag{15}$$

Thus, we obtain the following ODE from eq. (15) that:

$$\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} - v_1(\Theta)\Theta(\theta) = 0, \qquad \frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} - v_2(\Theta)\Theta(\theta) = 0 \tag{16a,b}$$

Furthermore, we directly write the solutions of eq. (15) from eqs. (16a) and (16b). Conveniently, we call this technology as the CPRM [8].

Traveling wave solutions for the non-linear heat transfer problem

In this section, the CPRM is used to solve the non-linear heat transfer equation.

From eqs. (5), (10a), and (10b) we have the non-linear ODE:

$$-\vartheta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} = \alpha\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2} - \beta\Theta(\theta) - \kappa\Theta^4(\theta) \tag{17}$$

which leads to:

$$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}+\frac{\vartheta}{\alpha}\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}-\frac{\beta}{\alpha}\Theta(\theta)-\frac{\kappa}{\alpha}\Theta^4(\theta)=0 \tag{18}$$

Following eq. (12), we give:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}-\sigma_1(\Theta)\right]\left[\frac{\mathrm{d}}{\mathrm{d}\theta}-\sigma_2(\Theta)\right]\Theta=0 \tag{19}$$

which yields the following equation in the factorized form:

$$\frac{\mathrm{d}^2\Theta(\theta)}{\mathrm{d}\theta^2}-\left[\frac{\mathrm{d}\sigma_1(\Theta)}{\mathrm{d}\Theta}\Theta+\sigma_1(\Theta)+\sigma_2(\Theta)\right]\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}+\sigma_1(\Theta)\sigma_2(\Theta)\Xi(\Theta)=0 \tag{20}$$

where

$$\frac{\mathrm{d}\sigma_1(\Theta)}{\mathrm{d}\Theta}\Theta+\sigma_1(\Theta)+\sigma_2(\Theta)=-\frac{\vartheta}{\alpha},\qquad \sigma_1(\Theta)\sigma_2(\Theta)=-\frac{\kappa}{\alpha}\left[\frac{\beta}{\kappa}+\Theta^3(\theta)\right] \tag{21a,b}$$

Thus, we have from eq. (21b) that:

$$\sigma_1(\Theta)=\frac{1}{H_{1,1}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\mp\sqrt{\frac{\beta}{\alpha}}\right],\qquad \sigma_2(\Theta)=H_{1,1}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\pm\sqrt{\frac{\beta}{\alpha}}\right] \tag{22a,b}$$

where $H_{1,1}$ is an unknown coefficient.

From eqs. (22a) and (22b), eq. (21a) can be written as:

$$-\frac{3i}{2H_{1,1}}\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)+\frac{1}{H_{1,1}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\mp\sqrt{\frac{\beta}{\alpha}}\right]+H_{1,1}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\pm\sqrt{\frac{\beta}{\alpha}}\right]=-\frac{\vartheta}{\alpha} \tag{23}$$

which leads to:

$$-\frac{5}{2H_{1,1}}+H_{1,1}=0,\qquad \pm\frac{1}{H_{1,1}}\mp H_{1,1}=\frac{\vartheta}{\sqrt{\alpha\beta}} \tag{24a,b}$$

From eq. (24a), we obtain:

$$H_{1,1}=\pm\sqrt{\frac{5}{2}} \tag{25}$$

which lead from eq. (19) to:

$$\sigma_1(\Theta)=\pm\sqrt{\frac{2}{5}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\mp\sqrt{\frac{\beta}{\alpha}}\right],\qquad \sigma_2(\Theta)=\pm\sqrt{\frac{5}{2}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\pm\sqrt{\frac{\beta}{\alpha}}\right] \tag{26a,b}$$

It follows from eqs. (26a) and (26b) that:

$$\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}-\left\{\pm\sqrt{\frac{2}{5}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\mp\sqrt{\frac{\beta}{\alpha}}\right]\right\}\Theta(\theta)=0 \tag{27}$$

$$\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}-\left\{\pm\sqrt{\frac{5}{2}}\left[i\sqrt{\frac{\kappa}{\alpha}}\Theta^{\frac{3}{2}}(\theta)\pm\sqrt{\frac{\beta}{\alpha}}\right]\right\}\Theta(\theta)=0 \tag{28}$$

By integrating eqs. (27) and (28), we obtain the following exact solutions, which are given by:

$$\Theta(\theta)=-\frac{\kappa}{4\beta}\left[\tanh\left(\frac{\beta\theta}{10\alpha}\right)+1\right]^2 \quad \text{or} \quad \Theta(\theta)=-\frac{\kappa}{4\beta}\left[\tanh\left(\frac{5\beta\theta}{8\alpha}\right)+1\right]^2 \tag{29a,b}$$

Thus, we obtain the exact traveling wave solutions, which are given by:

$$\Theta(\xi,\tau)=-\frac{\kappa}{4\beta}\left[\tanh\frac{\beta(\xi-\vartheta\tau)}{10\alpha}+1\right]^2 \tag{30a}$$

$$\Theta(\xi,\tau)=-\frac{\kappa}{4\beta}\left[\tanh\frac{5\beta(\xi-\vartheta\tau)}{8\alpha}+1\right]^2 \tag{30b}$$

where

$$\vartheta=\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{2}{5}}\right)\sqrt{\alpha\beta} \tag{30c}$$

with $\beta > 0$. The travelling wave solutions of eq. (4) for the different parameters are illustrated in figs. 2(a) and 2(b).

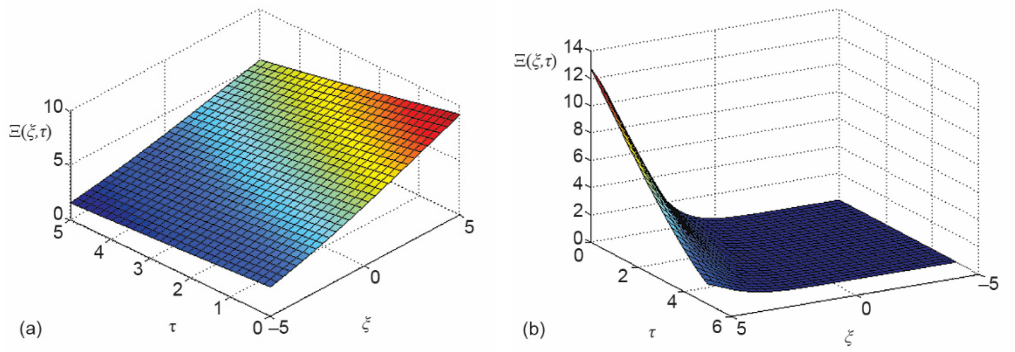

Figure 2. (a) The travelling wave solution (30a) for β = 1, α = 1, and κ = –1, (b) the travelling wave solution (30b) for β = 1, α = 1, and κ = –1

Conclusion

In the present work, a non-linear heat transfer equation was proposed for the first time. With the help of the CPRM, its exact traveling wave solutions were graphically illustrated. The CPRM to obtain the exact solutions were as an alternative technology proposed to find a class of the non-linear PDE in mathematical physics.

Acknowledgments

This work is supported by the State Key Research Development Program of the People's Republic of China (Grant No. 2016YFC0600705), the Natural Science Foundation of China (Grant No. 51323004), Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD2014), Sichuan Sci-Technology Support Program (2012FZ0124), National Natural Science Fundament of China (Grant No. 1137131361), Innovation Team of Jiangsu Province Hosted by China University of Mining and Technology (2014), and Key Discipline Construction by China University of Mining and Technology (Grant No. XZD 201602).

Nomenclature

α – heat diffusion coefficient, [$\mathrm{Wm^{-1}K^{-1}}$]
β – constant, [$\mathrm{1s^{-1}}$]
$\Theta(\xi, \tau)$ – exess temperature, [K]
κ – constant, [$\mathrm{K^3s^{-1}}$]
ξ – space co-ordinate, [m]
τ – time co-ordinate, [s]

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Paper submitted: May 12, 2016
Paper revised: June 30, 2016
Paper accepted: August 23, 2016