

A FRACTIONAL WHITHAM-BROER-KAUP EQUATION AND ITS POSSIBLE APPLICATION TO TSUNAMI PREVENTION

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A fractional Whitham-Broer-Kaup equation is suggested using He's fractional derivative to model solitary waves in shallow water in porous medium near a dam. A modification of the exp-function method, the generalized exponential rational function method, is adopted to elucidate the basic solution properties of the equation, revealing that the value of the fractional order can be used effectively to control the wave velocity, the wave height, and the wave morphology. This theoretical result can be used for possible tsunami prevention.

Key words: *He's fractional derivative, exp-function method, tsunami, generalized exponential rational function method, fractional Whitham-Broer-Kaup equation*

Introduction

Thermal science sees a large number of fractional models for various practical applications in discontinuous media [1-13]. It is still an open problem so far how to establish a fractional model for a practical problem, and to obtain its physical solution. Though fractional calculus becomes a hot topic in both mathematics and engineering, most works were focused on pure mathematics without possible applications. The main problem blocking its applications is the definition of fractional derivative. Previous definitions were of mathematical notations without any practical applications. The most used ones are Riemann-Liouville derivative and its various modifications, and many counter-examples were appeared in their applications [14]. This paper adopts He's fractional derivative [1 ,4, 7-13] defined:

$$D_t^\alpha u = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (s-t)^{n-\alpha-1} [u_0(s) - u(s)] ds \quad (1)$$

This definition has definitely physical understanding. If u is the solution for a discontinuous medium, u_0 is the solution for its continuous partner with same boundary and initial conditions.

Whitham-Broer-Kaup (WBK) equation has been widely studied in solitary theory arising in shallow water [15, 16], and its fractional partner describes shallow water in porous medium, which is used to absorb wave energy and prevent tsunami.

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The fractional WBK equation can be written in the form:

$$\begin{cases} D_t^\alpha u + u D_x^\alpha u + D_x^\alpha v + \beta D_x^{2\alpha} u = 0 \\ D_t^\alpha v + D_x^\alpha(uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0 \end{cases} \quad (2)$$

where α is the value of fractal dimensions of the porous medium placed near a dam, $0 < \alpha \leq 1$, $u = u(x, t)$ – the horizontal velocity, $v = v(x, t)$ – the height deviating from equilibrium position of liquid, and β and γ – the real constants that represent different diffusion powers. Fractional WBK equation with other fractional derivatives was studied in [15, 16], this paper adopts the definition given in eq. (1).

By the fractional complex transformation [1, 17-20]:

$$u(x, t) = U(\xi), \quad v(x, t) = V(\xi), \quad \text{and} \quad \xi = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ct^\alpha}{\Gamma(\alpha+1)} \quad (3)$$

where $k \neq 0$ and $c \neq 0$ are constants, eq. (2) reduces to the following form:

$$\begin{cases} cU' + kUU' + KV' + \beta k^2 U'' = 0 \\ cV' + k(U'V + UV') - \beta k^2 V'' + \gamma k^3 V''' = 0 \end{cases} \quad (4)$$

We suppose that the exact solution of eq. (4) can be expressed:

$$\begin{cases} U(\xi) = \sum_{i=0}^N \frac{a_i}{(1 \pm a^\xi)^i} \\ V(\xi) = \sum_{j=0}^M \frac{b_j}{(1 \pm a^\xi)^j} \end{cases} \quad (5)$$

where $a_i (i = 0, 1, \dots, N)$ and $b_j (j = 0, 1, \dots, M)$ are constants to be determined later.

Equation (5) is a modification of the exp-function method [21, 22], and it was named as the generalized exponential rational function method [23]. When we set $a^\xi = \exp(\xi \ln a)$, eq. (5) becomes a standard form required by the exp-function method.

By balancing the highest order derivative term and non-linear term in eq. (5), we receive $N+2 = 2N+1$, $N+3 = M+N+1$, then $N=1, M=2$. Thus eq. (5) becomes:

$$\begin{cases} U(\xi) = a_0 + \frac{a_1}{1 \pm a^\xi} \\ V(\xi) = b_0 + \frac{b_1}{1 \pm a^\xi} + \frac{b_2}{(1 \pm a^\xi)^2} \end{cases} \quad (6)$$

Substituting eq. (6) into eq. (4), and setting each coefficient of the same powers of a^ξ to be zero, we get a series of algebraic equations with respect to a_0, a_1, b_0, b_1 , and b_2 . Solving the algebraic equations by some a mathematical software, we have the following solutions.

When $4\beta^2 + 4\gamma > 0$, there are solutions in eqs. (7)-(10):

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2(4\beta^2 + 4\gamma)^{1/2} \ln a + 2c}{k} + \frac{(4\beta^2 + 4\gamma)^{1/2} (\ln a)k}{1+a^\xi} \\ v(x,t) = \frac{k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1+a^\xi} + \\ + \frac{-k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1+a^\xi)^2} \end{cases} \quad (7)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2(4\beta^2 + 4\gamma)^{1/2} \ln a + 2c}{k} + \frac{(4\beta^2 + 4\gamma)^{1/2} (\ln a)k}{1-a^\xi} \\ v(x,t) = \frac{k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1-a^\xi} + \\ + \frac{-k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1-a^\xi)^2} \end{cases} \quad (8)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{-k^2(4\beta^2 + 4\gamma)^{1/2} \ln a + 2c}{k} - \frac{(4\beta^2 + 4\gamma)^{1/2} (\ln a)k}{1+a^\xi} \\ v(x,t) = \frac{-k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1+a^\xi} + \\ + \frac{k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1+a^\xi)^2} \end{cases} \quad (9)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{-k^2(4\beta^2 + 4\gamma)^{1/2} \ln a + 2c}{k} - \frac{(4\beta^2 + 4\gamma)^{1/2} (\ln a)k}{1-a^\xi} \\ v(x,t) = \frac{-k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1-a^\xi} + \\ + \frac{k^2 \beta (4\beta^2 + 4\gamma)^{1/2} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1-a^\xi)^2} \end{cases} \quad (10)$$

when $4\beta^2 + 4\gamma < 0$, there are solutions in the form of eqs. (11)-(14):

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2 \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} \ln a + 2c}{k} + \frac{\{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)k}{1+a^\xi} \\ v(x,t) = \frac{k^2 \beta \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1+a^\xi} + \\ + \frac{-k^2 \beta \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1+a^\xi)^2} \end{cases} \quad (11)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2 \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} \ln a + 2c}{k} + \frac{\{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a) k}{1 + a^\xi} \\ v(x,t) = \frac{k^2 \beta \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1 + a^\xi} + \\ + \frac{-k^2 \beta \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1 + a^\xi)^2} \end{cases} \quad (12)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2 \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} \ln a + 2c}{k} + \frac{\{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a) k}{1 - a^\xi} \\ v(x,t) = \frac{k^2 \beta \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1 - a^\xi} + \\ + \frac{-k^2 \beta \{i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1 - a^\xi)^2} \end{cases} \quad (13)$$

$$\begin{cases} u(x,t) = -\frac{1}{2} \frac{k^2 \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} \ln a + 2c}{k} + \frac{\{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a) k}{1 - a^\xi} \\ v(x,t) = \frac{k^2 \beta \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 + 2(\ln a)^2 k^2 \beta^2 + 2\gamma k^2 (\ln a)^2}{1 - a^\xi} + \\ + \frac{-k^2 \beta \{-i[-(4\beta^2 + 4\gamma)]^{1/2}\} (\ln a)^2 - 2(\ln a)^2 k^2 \beta^2 - 2\gamma k^2 (\ln a)^2}{(1 - a^\xi)^2} \end{cases} \quad (14)$$

when $4\beta^2 + 4\gamma = 0$, there are solutions in the form:

$$\begin{cases} u(x,t) = -\frac{c}{k} \\ v(x,t) = 0 \end{cases} \quad (15)$$

where $i^2 = -1$, $\xi = kx^\alpha / \Gamma(\alpha + 1) + ct^\alpha / \Gamma(\alpha + 1)$.

The graphs of eq. (7) are illustrated in figs. 1(a), 1(b), 2(a), and 2(b) for $\beta = 1$, $\gamma = 1$, $k = 1$, $c = 1$, and $a = 10$. The graphs of eq. (8) are demonstrated in figs. 3(a), 3(b), 4(a), and 4(b), for $\beta = 2$, $\gamma = 3$, $k = 7$, $c = -4$, and $a = 10$. The graphs of eq. (9) are demonstrated in figs. 5(a), 5(b), 6(a), and 6(b) for $\beta = 1$, $\gamma = 1$, $k = 1$, $c = 1$, and $a = 10$. The graphs of eq. (10) are demonstrated in figs. 7(a), 7(b), 8(a), and 8(b), for $\beta = 2$, $\gamma = 3$, $k = 7$, $c = -4$, and $a = 10$. The graphs of eq. (11) are demonstrated in figs. 9(a), 9(b), 10(a), and 10(b) for $\beta = 1$, $\gamma = -2$, $k = 1$, $c = 1$, and $a = 3$, other situations omitted. From the given illustrations, we can find that the velocity of a solitary wave can suddenly increase to its maximum, see fig. 1(a), this means the solitary wave reaches its maximal kinetic energy which can make great catastrophe. We also find that the height of solitary wave reaches its maximum, which will result in a possible tsunami.

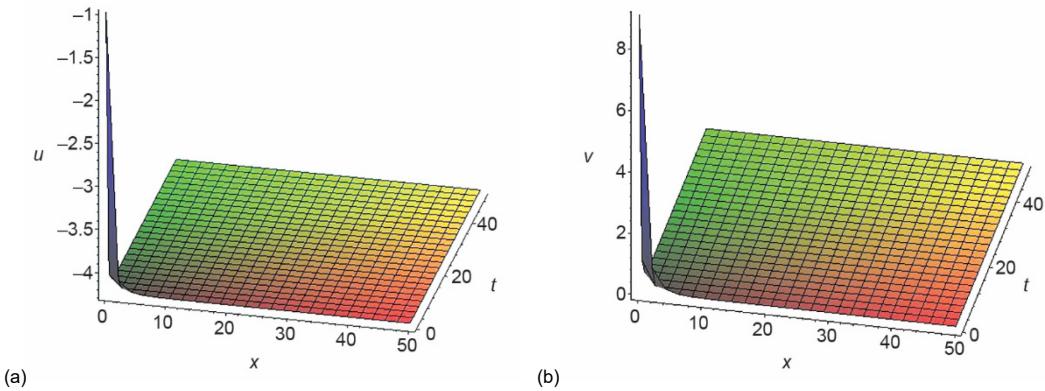


Figure 1. Solitary waves of the fractional WBK eq. (7); (a) u , $\alpha = 1/2$, (b) v , $\alpha = 1/2$
(for color image see journal web site)

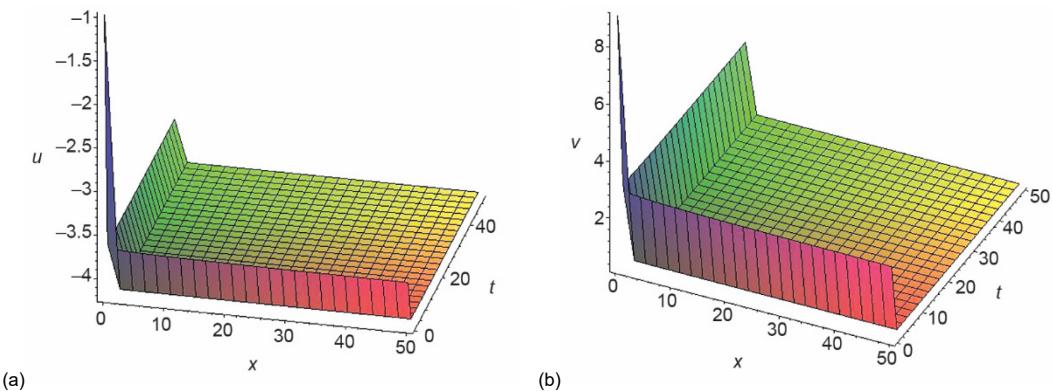


Figure 2. Solitary waves of the fractional WBK eq. (7); (a) u , $\alpha = 1/100$, (b) v , $\alpha = 1/100$
(for color image see journal web site)

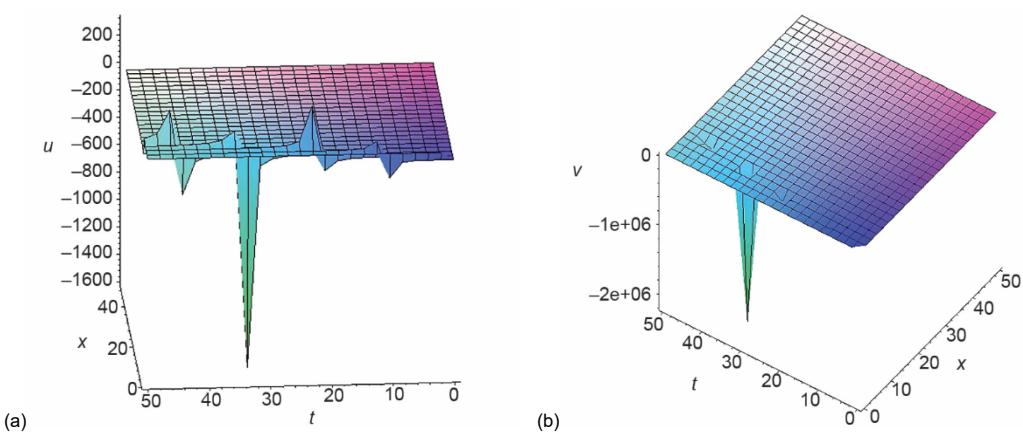


Figure 3. Solitary waves of the fractional WBK eq. (8); (a) u , $\alpha = 1/3$, (b) v , $\alpha = 1/3$

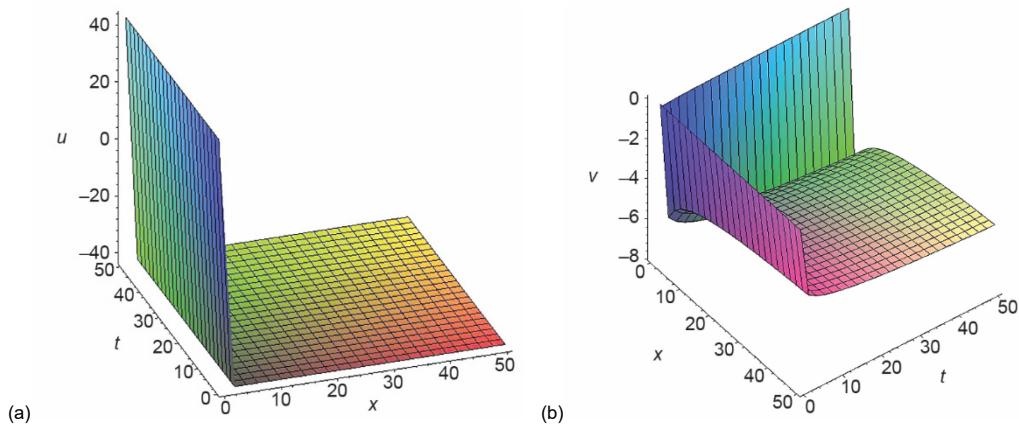


Figure 4. Solitary waves of the fractional WBK eq. (8); (a) u , $\alpha = 1/100$, (b) v , $\alpha = 1/100$
 (for color image see journal web site)

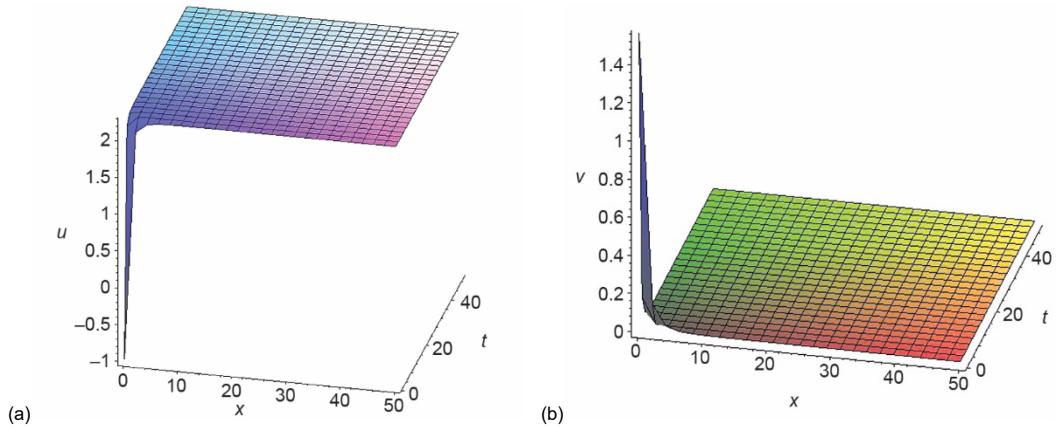


Figure 5. Solitary waves of the fractional WBK eq. (9); (a) u , $\alpha = 1/2$, (b) v , $\alpha = 1/2$
 (for color image see journal web site)

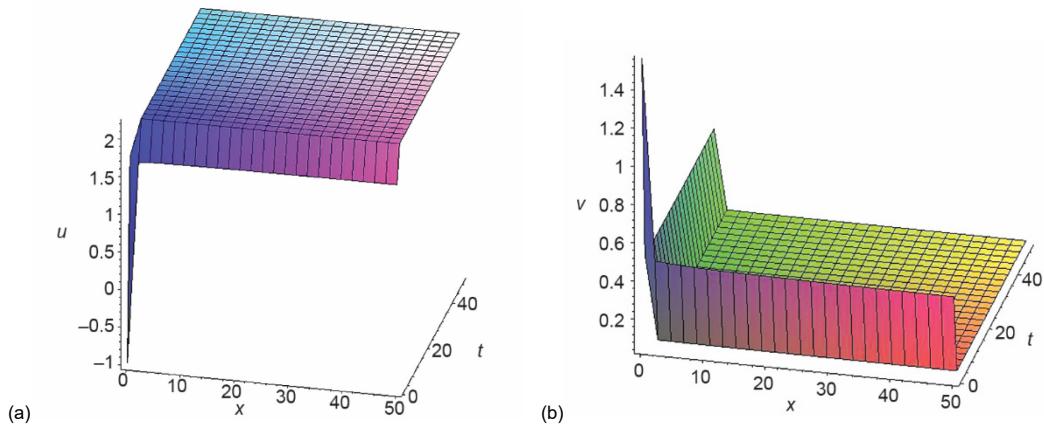


Figure 6. Solitary waves of the fractional WBK eq. (9); (a) u , $\alpha = 1/100$, (b) v , $\alpha = 1/100$

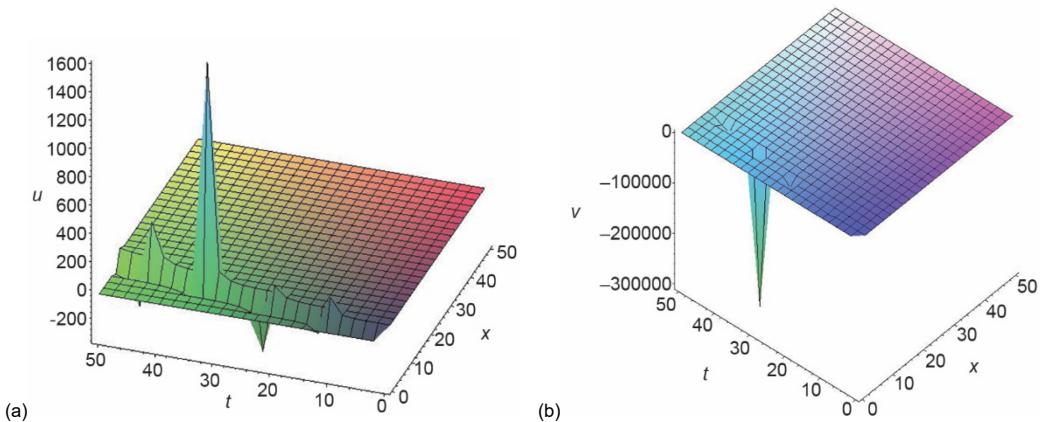


Figure 7. Solitary waves of the fractional WBK eq. (10); (a) u , $\alpha = 1/3$, (b) v , $\alpha = 1/3$
(for color image see journal web site)

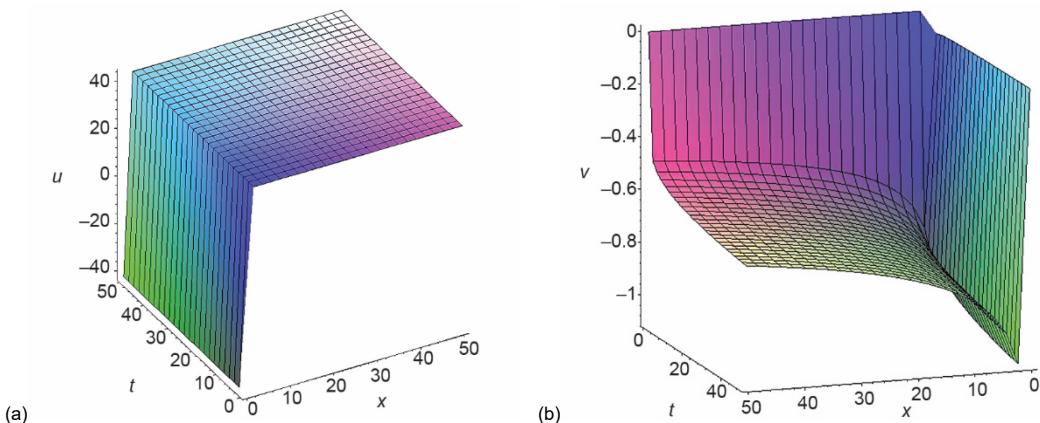


Figure 8. Solitary waves of the fractional WBK eq. (10); (a) u , $\alpha = 1/100$, (b) v , $\alpha = 1/100$
(for color image see journal web site)

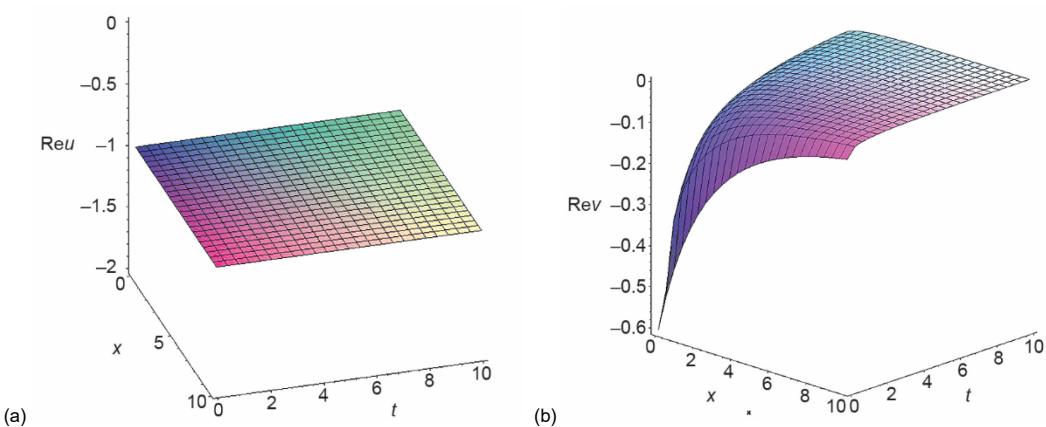


Figure 9. Solitary waves of the fractional WBK eq. (11); (a) $\text{Re } u$, $\alpha = 1/2$, (b) $\text{Re } v$, $\alpha = 1/2$

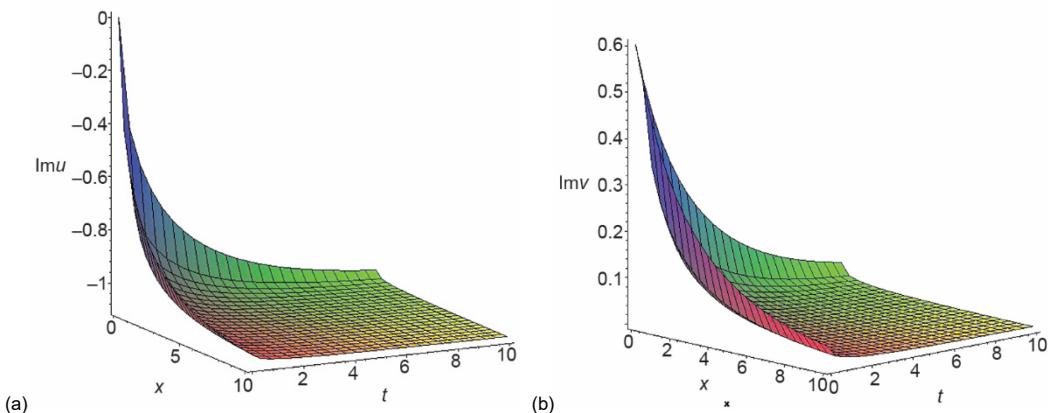


Figure 10. Solitary waves of the fractional WBK eq. (11); (a) $\text{Im } u$, $\alpha = 1/2$, (b) $\text{Im } v$, $\alpha = 1/2$

Discussion and conclusion

This paper gives an analytical approach to fractional differential equations. The complex variable ξ is a function of the order of the fractional derivative:

$$\xi(\alpha) = \frac{kx^\alpha}{\Gamma(\alpha+1)} + \frac{ct^\alpha}{\Gamma(\alpha+1)} \quad (16)$$

When $\alpha = 1$, we obtain solutions for the classic WBK equation, u_0 . The value of α strongly depends upon the fractal dimensions of porous structure of the porous medium for tsunami prevention. The paper reveals that solitary wave velocity and its height can be controlled by mainly the value of α , the porosity of porous medium, which can be stone walls or randomly placed balloons.

We conclude in this paper that the solitary waves can be effectively controlled by porous medium in shallow water, as a result, a dam can be well protected and tsunami can be avoided by suitable adjusting the porosity of porous medium in shallow water.

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Nomenclature

t	– time co-ordinate, [s]
$u(x, t)$	– velocity, [ms^{-1}]
x	– space co-ordinate, [m]

Greek symbols

α	– fractional order, [-]
β, γ	– constants, [-]
$v(x, t)$	– height deviation, [m]

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