AN EXPLANATION OF LOCAL FRACTIONAL VARIATIONAL ITERATION METHOD AND ITS APPLICATION TO LOCAL FRACTIONAL MODIFIED KORTEWEG-DE VRIES EQUATION

by

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The variational iteration method was originally proposed to solve non-linear problems of differential equations, this paper shows that it is also a powerful mathematical tool to local fractional differential equations. Two local fractional modified Korteweg-de Vries equations are used as examples to reveal the simple solution process.

Key words: *modified Korteweg-de Vries equation, local fractional derivative, local fractional variational iteration method, series solution*

Introduction

The Korteweg-de Vries (KdV) equation is a well-known non-linear equation for solitary theory. It arises in modelling waves on shallow water surface:

$$\frac{\partial\phi(\mu,\tau)}{\partial\tau}+6\phi(\mu,\tau)\frac{\partial\phi(\mu,\tau)}{\partial\mu}+\frac{\partial^3\phi(\mu,\tau)}{\partial\mu^3}=0 \tag{1}$$

Equation (1) is derived by the assumption of continuum of water. In the case that water is involved in porous medium, the continuum mechanics is forbidden, and fractional partner has to be adopted [1]. Two modifications of KdV equation with a local fractional derivative are considered for fractal space and fractal time [2, 3], which are, respectively,

$$\begin{cases}\dfrac{\partial^\varepsilon\phi(\mu,\tau)}{\partial\tau^\varepsilon}+\dfrac{\partial^{3\varepsilon}\phi(\mu,\tau)}{\partial\mu^{3\varepsilon}}=0\\ \phi(\mu,0)=E_\varepsilon(\mu^\varepsilon)\end{cases} \tag{2}$$

and

$$\begin{cases}\dfrac{\partial^\varepsilon\Phi(\mu,\tau)}{\partial\tau^\varepsilon}+\dfrac{\partial^\varepsilon\Phi(\mu,\tau)}{\partial\mu^\varepsilon}+\dfrac{\partial^{3\varepsilon}\Phi(\mu,\tau)}{\partial\mu^{3\varepsilon}}=0\\ \Phi(\mu,0)=-E_\varepsilon(\mu^\varepsilon)\end{cases} \tag{3}$$

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where $0 < \varepsilon < 1$, $\partial^\varepsilon/\partial\tau^\varepsilon$, $\partial^\varepsilon/\partial\mu^\varepsilon$, and $\partial^{3\varepsilon}/\partial\mu^{3\varepsilon}$ are the local fractional partial derivatives, $\phi(\mu,\tau)$ and $\Phi(\mu,\tau)$ are the non-differentiable wave functions, and $E_\varepsilon(\mu^\varepsilon)$ is the Mittag-Leffler function defined on Cantor sets [2]. Recently analytical approaches to fractional differential equations have been caught much attention, for examples, the homotopy perturbation method [1, 4], the exp-function method [1, 5, 6], the sub-equation method [7], the homotopy analysis method [8], method of separation of variables [9], series expansion solution [10], Laplace series method [11], and fractional complex transform [11-14], among which the variational iteration method [15-22] is the most effective and simple method for local fractional calculus.

Local fractional variational iteration method

The variational iteration method (VIM) [15] is a powerful mathematical tool to all kinds of non-linear problems, and it was successfully extended to some fractional differential equations [16-22]. This paper shows that it is also an effective method for local fractional differential equations.

The local fractional variational iteration algorithm [1] for eq. (2) can be written:

$$\phi_{n+1}(\mu,\tau) = \phi_n(\mu,\tau) + \int_0^\tau \lambda\left[\frac{\partial^\varepsilon \phi_n(\mu,\xi)}{\partial\xi^\varepsilon} + \frac{\partial^{3\varepsilon}\phi_n(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon \tag{4}$$

where λ is the general Lagrange multiplier [23], which can be identified by the fractional variational theory [1], which reads:

$$\lambda = -\frac{1}{\Gamma(1+\varepsilon)} \tag{5}$$

The following iteration formulation is, therefore, obtained:

$$\phi_{n+1}(\mu,\tau) = \phi_n(\mu,\tau) - \frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau \left[\frac{\partial^\varepsilon \phi_n(\mu,\xi)}{\partial\xi^\varepsilon} + \frac{\partial^{3\varepsilon}\phi_n(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon \tag{6}$$

We begin with:

$$\phi_0(\mu,\tau) = \phi_0(\mu,0) = E_\varepsilon(\mu^\varepsilon) \tag{7}$$

and we obtain the following approximations:

$$\phi_1(\mu,\tau) = \phi_0(\mu,\tau) - \frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau \left[\frac{\partial^\varepsilon \phi_0(\mu,\xi)}{\partial\xi^\varepsilon} + \frac{\partial^{3\varepsilon}\phi_0(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon = E_\varepsilon(\mu^\varepsilon)\left[1 - \frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)}\right] \tag{8}$$

$$\phi_2(\mu,\tau) = \phi_1(\mu,\tau) - \frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau \left[\frac{\partial^\varepsilon \phi_1(\mu,\xi)}{\partial\xi^\varepsilon} + \frac{\partial^{3\varepsilon}\phi_1(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon =$$

$$= E_\varepsilon(\mu^\varepsilon)\left[1 - \frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{\tau^{2\varepsilon}}{\Gamma(1+2\varepsilon)}\right] \tag{9}$$

The iteration continues with ease, and a series solution is obtained, which is:

$$\phi_n(\mu,\tau)=\phi_{n-1}(\mu,\tau)-\frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau\left[\frac{\partial^\varepsilon\phi_{n-1}(\mu,\xi)}{\partial\xi^\varepsilon}+\frac{\partial^{3\varepsilon}\phi_{n-1}(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon=$$

$$=E_\varepsilon(\mu^\varepsilon)\left[1+(-1)\frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)}+(-1)^2\frac{\tau^{2\varepsilon}}{\Gamma(1+2\varepsilon)}+\cdots+(-1)^n\frac{\tau^{n\varepsilon}}{\Gamma(1+n\varepsilon)}\right] \tag{10}$$

In case n tends to infinite, we obtain an exact solution:

$$\phi(\mu,\tau)=\lim_{n\to\infty}\phi_n(\mu,\tau)=E_\varepsilon(\mu^\varepsilon)\sum_{k=0}^{\infty}(-1)^k\frac{\tau^{k\varepsilon}}{\Gamma(1+k\varepsilon)}=E_\varepsilon(\mu^\varepsilon)E_\varepsilon(-\tau^\varepsilon) \tag{11}$$

In the same way, the local fractional variational iteration algorithm of eq. (3) can be written in the form:

$$\Phi_{n+1}(\mu,\tau)=\Phi_n(\mu,\tau)+\int_0^\tau\lambda\left[\frac{\partial^\varepsilon\Phi_n(\mu,\xi)}{\partial\xi^\varepsilon}+\frac{\partial^\varepsilon\Phi_n(\mu,\xi)}{\partial\mu^\varepsilon}+\frac{\partial^{3\varepsilon}\Phi_n(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon \tag{12}$$

where λ is identified by the same way as that in eq. (4). The iteration formulation is:

$$\Phi_{n+1}(\mu,\tau)=\Phi_n(\mu,\tau)-\frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau\left[\frac{\partial^\varepsilon\Phi_n(\mu,\xi)}{\partial\xi^\varepsilon}+\frac{\partial^\varepsilon\Phi_n(\mu,\xi)}{\partial\mu^\varepsilon}+\frac{\partial^{3\varepsilon}\Phi_n(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon \tag{13}$$

We begin with:

$$\Phi_0(\mu,\tau)=-E_\varepsilon(\mu^\varepsilon) \tag{14}$$

and a series solution is obtained, which is:

$$\Phi_n(\mu,\tau)=\Phi_{n-1}(\mu,\tau)-\frac{1}{\Gamma(1+\varepsilon)}\int_0^\tau\left[\frac{\partial^\varepsilon\Phi_{n-1}(\mu,\xi)}{\partial\xi^\varepsilon}+\frac{\partial^\varepsilon\Phi_{n-1}(\mu,\xi)}{\partial\mu^\varepsilon}+\frac{\partial^{3\varepsilon}\Phi_{n-1}(\mu,\xi)}{\partial\mu^{3\varepsilon}}\right](\mathrm{d}\xi)^\varepsilon=$$

$$=-E_\varepsilon(\mu^\varepsilon)\left[1+(-2)\frac{\tau^\varepsilon}{\Gamma(1+\varepsilon)}+(-2)^2\frac{\tau^{2\varepsilon}}{\Gamma(1+2\varepsilon)}+\cdots+(-2)^n\frac{\tau^{n\varepsilon}}{\Gamma(1+n\varepsilon)}\right] \tag{15}$$

Consequently, the local fractional series solution of eq. (3) is expressed as:

$$\Phi(\mu,\tau)=\lim_{n\to\infty}\Phi_n(\mu,\tau)=-E_\varepsilon(\mu^\varepsilon)\sum_{k=0}^{\infty}(-2)^k\frac{\tau^{k\varepsilon}}{\Gamma(1+k\varepsilon)}=-E_\varepsilon(\mu^\varepsilon)E_\varepsilon(-2\tau^\varepsilon) \tag{16}$$

The obtained solutions, eqs. (11) and (16), are illustrated in figs. 1 and 2 for $k=0,1,\cdots,500,$ respectively. These solutions are expressed in Mittag-Leffler function [2, 23].

Discussion and conclusion

The VIM leads to a solution using Mittag-Leffler function. The properties of Mittag-Leffler function were systematically studied in [23], according to its basic properties; the obtained solutions illustrated in figs. 1 and 2 are actually non-smooth everywhere, and the wave

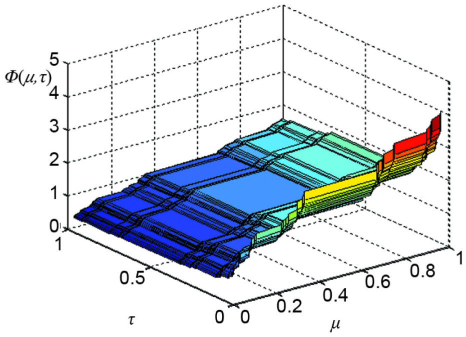

Figure 1. The series solution of eq. (2)
(*for color image see journal web site*)

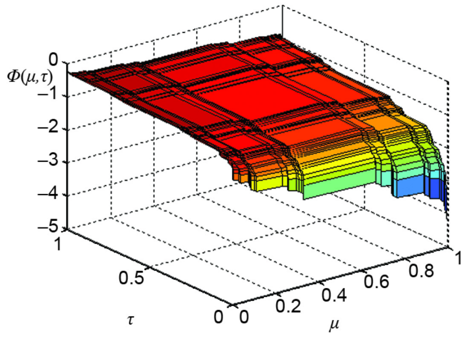

Figure 2. The series solution of eq. (3)
(*for color image see journal web site*)

morphology can be controlled by suitable adjustment of the parameters involved in the Mittag-Leffler function.

This paper shows that the VIM is also valid for local fractional differential equation, and it is extremely suitable for an initial problem. Generally, a series solution is obtained, which is useful to analyze the dynamics of solitary waves.

Acknowledgments

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Nomenclature

μ – space co-ordinate, [m]
τ – time co-ordinate, [s]
ε – fractional order, [–]

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