MIXED CONVECTION FLOW OF A NANOFLUID PAST A NON-LINEARLY STRETCHING WALL

by

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This paper deals with the boundary-layer mixed convective flow of a viscous nanofluid past a vertical wall stretching with non-linear velocity. The governing equations are transformed into self similar ordinary differential equations using appropriate transformation. Using group theoretic method it is shown that the similarity solutions are possible only for the non-linear stretching velocity having specific form. Numerical solution of the coupled governing equations is obtained using Keller Box method. Correlation expression of reduced Nusselt and Sherwood numbers are obtained by performing linear regression on the data obtained from numerical results. The authenticity of these results is established by calculating the percentage error between the numerical results and correlation expression which is observed to be less than 5%. Effects of Brownian and thermophoretic diffusions and nanoparticles concentration flux on the Nusselt and Sherwood numbers are discussed.

Key words*: mixed convection flow, non-linear stretching, nanofluid, correlation expression*

Introduction

The model presented by Buongiorno [1] is considered in this article to examine the mixed convective flow and heat transfer of nanofluid past a non-linearly stretching sheet. The problem of natural convection past a vertical plate was first studied theoretically and experimentally by Pohlhausen *et al.* [2]. Merkin and Pop [3] used a similarity transformation to analyze mixed convection boundary-layer flow over a vertical semi-infinite plate. Steinruck [4] found a new similarity solution of mixed convection flow along a vertical plate. Kuznetsov and Nield [5, 6] studied the problems of natural convection for nanofluid.

Considering the impact of stretching and heating/cooling of surface on the quality of the finished product of real processes, the modeling of such processes is undertaken with the help of different stretching velocities and surface temperature distributions. These stretching are linear [7], polynomial [8], and exponential [9]. A lot of research has been conducted on stretching phenomena in Newtonian and non-Newtonian fluids and has been widely and extensively quoted. Recently, the stretching of plate in nanofluid has been a focus of study due to its applications. Bachok *et al.* [10] studied the flow of nanofluid over a moving surface. Khan and Pop

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[11] investigated numerically the problem of laminar flow of nanofluid due to stretching of the surface. Rana and Bhargava [12] investigated the nanofluid flow which results from the sheet stretching non-linearly. Matin and Jahangiri [13] presented numerical solution using Keller-box method for the forced convection MHD flow of nanofluid over a permeable stretching plate with viscous dissipation. Hayat *et al.* [14] studied the MHD flow of viscous nanofluid caused by a permeable exponentially stretching surface. In [15] Hayat *et al.* addresses the mixed convection flow of Casson nanofluid over a stretching surface in presence of thermal radiation, heat source/sink and first order chemical reaction. Ahmad *et al.* [16] explores the boundary-layer flow and heat transfer of a viscous nanofluid bounded by a hyperbolically stretching sheet. Recently, Ahmad [17] studied the flow of a classical non-Newtonian fluid, Reiner-Philippoff fluid, in the presence of nanoparticles over a non-linear stretching sheet.

Why this paper

In a recent paper, Xu and Pop [18] solved a problem of mixed convection flow of a nanofluid over a linearly stretching surface in a uniform free stream velocity. They introduced a similarity transformation which converts the governing equations into a set of self-similar non-linear ODE (suitable for numerical solution). Using group theoretic method, we show that that the similarity transformation used in [18] for uniform free stream is unique. This means to say that in the absence of free stream velocity, uniform or otherwise, these equations do not admit any similarity transformation for linear stretching. The question arises; is this condition hold for non-linear stretching as well. In this paper, we provide an answer to this question and show that there exists a non-linear stretching of the form $x^{1/2}$ for which the similarity transformations are possible when there is no free stream.

After having found the similarity transformations by group theoretic method; we solve the problem of mixed convection flow of nanofluid over a non-linear ($x^{1/2}$) stretching surface in the absence of free stream. We convert the governing equations into self-similar non-linear ODE using the similarity so obtained. The coupled non-linear resulting equations are solved numerically. The telling points of this study are: (a) the full range of possible similarity transformations for the flow of nanofluid past linear and non-linear stretching of a vertical plate has been presented by group theoretic method; this will help the scientists and engineers alike, to discern what to focus and what not to focus on; while attempting the new problems using similarity transformations, (b) a self-similar boundary value problem for the mixed convection flow of nanofluid past non-linear stretching surface is formulated and the solution is presented for the first time, and (c) finding the correlation expression of reduced Nusselt and Sherwood numbers using linear regression on the numerical results.

The problem formulation

Vertical stretching sheet at $y = 0$ with x-axis aligned vertically upward has been considered. The sheet is stretching with non-linear velocity in x-direction in a viscous based nanofluid. At the sheet the temperature, T, and the nanoparticle fraction, ϕ, take the constant values T_w and ϕ_w, respectively. The ambient values of T and ϕ are given as T_∞ and ϕ_∞.

Employing the Oberbeck–Boussinesq approximation and the assumption that the nanoparticle concentration is dilute, the boundary-layer equations for the conservation of total mass, momentum, thermal energy, and nanoparticles are written:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{1}$$

$$\rho_f\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu\frac{\partial^2 u}{\partial y^2}+\left(1-\phi_\infty\right)\rho_f g\beta\left(T-T_\infty\right)-\left(\rho_\mathrm{p}-\rho_f\right)\mathrm{g}\left(\phi-\phi_\infty\right) \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial T}{\partial y}\frac{\partial \phi}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] \tag{3}$$

$$u\frac{\partial \phi}{\partial x}+v\frac{\partial \phi}{\partial y}=D_B\frac{\partial^2 \phi}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

where u and v are the velocity components in x- and y-direction, respectively, ρ_f – the density of the base fluid, ρ_p is the density of particles, and μ, α, and β are the viscosity, thermal conductivity, and volumetric volume expansion coefficient of the nanofluid, respectively. Also, in eqs. (1)-(4), g is the gravitational acceleration, D_B – the Brownian diffusion coefficient, D_T – the thermophoretic diffusion coefficient, and α and τ are defined:

$$\alpha=\frac{k}{(\rho c)_f},\qquad \tau=\frac{(\rho c)_\mathrm{p}}{(\rho c)_f}$$

The appropriate boundary conditions of the problem are:

$$\begin{gathered} u(x,y)=U(x),\quad v(x,y)=0,\ T=T_w,\quad \phi=\phi_w\quad \text{at}\quad y=0\\ u(x,y)\to 0,\quad T=T_\infty,\quad \phi=\phi_\infty\quad \text{as}\quad y\to\infty \end{gathered} \tag{5}$$

Introducing the non-dimensional parameters

$$\begin{gathered} \bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{L}\sqrt{\frac{\rho_f U_0 L}{\mu}},\quad \bar{T}=\frac{T-T_\infty}{T_\mathrm{w}-T_\infty},\quad \bar{\phi}=\frac{\phi-\phi_\infty}{\phi_\mathrm{w}-\phi_\infty}\\ \bar{u}=\frac{u}{U_0},\quad \bar{v}=\frac{v}{U_0}\sqrt{\frac{\rho_f U_0 L}{\mu}},\quad \bar{v}_\mathrm{w}=\frac{v_\mathrm{w}}{U_0}\sqrt{\frac{\rho_f U_0 L}{\mu}} \end{gathered} \tag{6}$$

the boundary value problem of eqs. (1)-(5) and obtain the dimensionless equations of the form:

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0 \tag{7}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}=\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}+\lambda\bar{T}-N_r\bar{\phi} \tag{8}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}=\frac{1}{\mathrm{Pr}}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}+N_b\frac{\partial \bar{T}}{\partial \bar{y}}\frac{\partial \bar{\phi}}{\partial \bar{y}}+N_t\left(\frac{\partial \bar{T}}{\partial \bar{y}}\right)^2 \tag{9}$$

$$\bar{u}\frac{\partial \bar{\phi}}{\partial \bar{x}}+v\frac{\partial \bar{\phi}}{\partial \bar{y}}=\frac{1}{\mathrm{Le}}\frac{\partial^2 \bar{\phi}}{\partial \bar{y}^2}+\frac{1}{\mathrm{Le}}\frac{N_t}{N_b}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\sqrt{2} \tag{10}$$

$$\begin{gathered} \bar{u}=U(\bar{x}),\quad \bar{v}=0,\quad \bar{T}=\frac{T_\mathrm{w}-T_\infty}{T_0},\quad \bar{\phi}=\frac{\phi_\mathrm{w}-\phi_\infty}{\phi_0}\quad \text{at}\quad \bar{y}=0\\ u(\bar{x},\bar{y})\to 0,\quad \bar{T}=0,\quad \bar{\phi}=0\quad \text{as}\quad \bar{y}\to\infty \end{gathered} \tag{11}$$

The dimensionless parameters in the previous equations are given:

$$\lambda = \frac{L\Delta T}{U_0^{\,2}}(1-\phi_\infty)\mathrm{g}\beta, \quad N_r = \frac{L\Delta\phi}{U_0^{\,2}}\left(\frac{\rho_\mathrm{p}-\rho_f}{\rho_f}\right)\mathrm{g}, \quad N_b = \frac{\tau D_B \Delta\phi}{\nu}$$

$$N_t = N_b = \frac{\tau D_T \Delta T}{\nu T_\infty}, \quad \nu = \frac{\mu}{\rho_f}, \quad \mathrm{Pr} = \frac{\nu}{\alpha}, \quad \mathrm{Le} = \frac{\nu}{D_B}$$

where λ is the Richardson number, N_r – the nanoparticles concentration flux parameter, N_b – Brownian motion parameter, N_t – thermophoresis parameter, Pr – the Prandtl's number, ν – the kinematic viscosity of the fluid, and Le – the Lewis number.

It is worth mentioning that rather than taking any specific form we have assumed that the wall stretching velocity is generalized function of x. However, using group theoretic method (given below) we reach at the conclusion that similarity transformation for the modeled problem is possible only if we take stretching velocity $U(x) = x^{1/2}$.

The scaling group of transformations

The similarity transformations are obtained by introducing the following scaling group of transformations [19-21]:

$$\Gamma : x^* = \bar{x}\mathrm{e}^{\varepsilon a}, \quad y^* = \bar{y}\mathrm{e}^{\varepsilon b}, \quad u^* = \bar{u}\mathrm{e}^{\varepsilon c}, \quad v^* = \bar{v}\mathrm{e}^{\varepsilon d}, \quad T^* = \bar{T}\mathrm{e}^{\varepsilon e}, \quad \phi^* = \bar{\phi}\mathrm{e}^{\varepsilon p}, \quad U^* = \bar{U}\mathrm{e}^{\varepsilon q} \quad (12)$$

where ε is a small parameter and a, b, c, d, e, p, and q are some constants to be determined. The point transformation, Γ, transforms the co-ordinates (x, y, u, v, T, ϕ, U) to the new co-ordinates ($x^*, y^*, u^*, v^*, T^*, \phi^*, U^*$).

Substituting eq. (12) in eqs. (7)-(11), we get:

$$\frac{\partial u^*}{\partial x^*} + \mathrm{e}^{\varepsilon(-a+b+c-d)}\frac{\partial v^*}{\partial y^*} = 0 \quad (13)$$

$$u^*\frac{\partial u^*}{\partial x^*} + \mathrm{e}^{\varepsilon(-a+b+c-d)}v^*\frac{\partial u^*}{\partial y^*} = \mathrm{e}^{\varepsilon(-a+2b+c)}\frac{\partial^2 u^*}{\partial y^{*2}} + \lambda\mathrm{e}^{\varepsilon(-a+2c-e)}T^* - N_r\mathrm{e}^{\varepsilon(-a+2c-p)}\phi^* \quad (14)$$

$$u^*\frac{\partial T^*}{\partial x^*} + \mathrm{e}^{\varepsilon(-a+b+c-d)}v^*\frac{\partial T^*}{\partial y^*} = \mathrm{e}^{\varepsilon(-a+2b+c)}\frac{1}{\mathrm{Pr}}\frac{\partial^2 T^*}{\partial y^{*2}} + N_b\mathrm{e}^{\varepsilon(-a+2b+c-p)}\frac{\partial T^*}{\partial y^*}\frac{\partial \phi^*}{\partial y^*} + N_t\mathrm{e}^{\varepsilon(-a+2b+c-e)}\left(\frac{\partial T^*}{\partial y^*}\right)^2 \quad (15)$$

$$u^*\frac{\partial \phi^*}{\partial x^*} + \mathrm{e}^{\varepsilon(-a+b+c-d)}v^*\frac{\partial \phi^*}{\partial y^*} = \frac{1}{\mathrm{Le}}\mathrm{e}^{\varepsilon(-a+2b+c)}\frac{\partial^2 \phi^*}{\partial y^{*2}} + \frac{1}{\mathrm{Le}}\frac{N_b}{N_t}\mathrm{e}^{\varepsilon(-a+2b+c-e+p)}\frac{\partial^2 T^*}{\partial y^{*2}} \quad (16)$$

$$\begin{gathered} u^*(x^*,0) = \mathrm{e}^{\varepsilon(c-q)}U^*(x^*), \quad v^*(x^*,0) = 0, \quad \mathrm{e}^{-\varepsilon e}T^*(x^*,0) = 1, \quad \mathrm{e}^{-\varepsilon p}\phi^*(x^*,0) = 1 \\ u^*(x^*,\infty) = 0, \quad T^*(x^*,\infty) = 0, \quad \phi^*(x^*,\infty) = 0 \end{gathered} \quad (17)$$

The previous system of equations becomes invariant under Γ, when the parameters defined in eq. (12) satisfy the following relations:

$$\begin{gathered} -a+b+c-d = 0, \quad -a+2b+c = 0, \quad -a+2c-e = 0, \quad -a+2c-p = 0 \\ -a+2b+c-p = 0, \quad -a+2b+c-e = 0, \quad -a+2b+c-e+p = 0 \\ c-q = 0, \quad e = 0, \quad q = 0 \end{gathered} \quad (18)$$

The solution of these equations in terms of a is given:

$$b = -d = \frac{a}{4}, \quad c = \frac{a}{2} \quad (19)$$

For these relations of constants, the scaling transformation (12) reduced to the following form:

$$\begin{aligned}\Gamma:\ x^* = \bar{x}e^{\varepsilon a},\quad y^* = \bar{y}e^{\varepsilon a/4},\quad u^* = \bar{u}e^{\varepsilon a/2},\quad v^* = \bar{v}e^{-\varepsilon a/4}\\ T^* = \bar{T},\quad \phi^* = \bar{\phi},\quad U^* = \bar{U}e^{\varepsilon a/2}\end{aligned} \tag{20}$$

Expanding the exponentials in (20) using Taylor series expansion, up to order, ε, we get:

$$\Gamma:\begin{cases} x^* - \bar{x} = \varepsilon a\,\bar{x},\quad y^* - \bar{y} = \dfrac{\varepsilon a}{4}\bar{y},\quad u^* - \bar{u} = \dfrac{\varepsilon a}{2}\bar{u},\quad v^* - \bar{v} = -\dfrac{\varepsilon a}{4}\bar{v} \\ T^* - \bar{T} = 0,\quad \phi^* - \bar{\phi} = 0,\quad U^* - \bar{U} = \dfrac{\varepsilon a}{2}\bar{U}\end{cases} \tag{21}$$

The characteristic equation is obtained by denoting the differences between the new and the original variables as differentials and equating each term, gives:

$$\frac{\mathrm{d}\bar{x}}{a\bar{x}} = \frac{\mathrm{d}\bar{y}}{\frac{a}{4}\bar{y}} = \frac{\mathrm{d}\bar{u}}{\frac{a}{2}\bar{u}} = \frac{\mathrm{d}\bar{v}}{-\frac{a}{4}\bar{v}} = \frac{\mathrm{d}\bar{T}}{0} = \frac{\mathrm{d}\bar{\phi}}{0} = \frac{\mathrm{d}\bar{U}}{\frac{a}{2}\bar{U}} \tag{22}$$

For the simplicity, we take $a = 1$. Now solving the previous system of equations, the following similarity transformations are obtained:

$$\eta = \frac{y}{C_1 x^{1/4}},\ \bar{u} = \bar{x}^{1/2} f'(\eta),\ \bar{v} = \bar{x}^{-1/4} h(\eta),\ \bar{T} = \theta(\eta),\ \bar{\phi} = s(\eta),\ U(\bar{x}) = C_2 \bar{x}^{1/2} \tag{23}$$

where C_1 and C_2 are some constants. The equation of continuity gives the actual form of $h(\eta)$. It is clear from eq. (23) that the form of stretching velocity is non-linear, $x^{1/2}$. Thus using group theoretic method, we can claim that for the mixed convection flow of nanofluid past a stretching surface the similarity transformation is possible only for $U(x) = x^{1/2}$. This observation is given in the literature for the first time.

Here we assume $C_1 = 3^{1/2}/2$ and $C_2 = 1$, for the sake of simplicity. Using the continuity equation, we can define stream function, ψ, such that:

$$\bar{u} = \frac{\partial \psi}{\partial \bar{y}},\quad \bar{v} = -\frac{\partial \psi}{\partial \bar{x}} \tag{24}$$

Equation (23) together with eq. (24) gives:

$$\psi = \frac{2}{\sqrt{3}} \bar{x}^{3/4} f(\eta) \tag{25}$$

Note that the two forms for v will be consistent if:

$$h(\eta) = -\frac{\sqrt{3}}{2}\left(f - \frac{\eta}{3} f'\right) \tag{26}$$

Using the similarity transformations and variables given by eq. (23), the eqs. (7)-(11) are conveniently transformed into ODE:

$$f''' + ff'' - \frac{2}{3} f'^2 + \frac{4}{3}\lambda\theta - \frac{4}{3} N_r s = 0 \tag{27}$$

$$\theta'' + \Pr f\theta' + N_b \theta' s' + N_t \theta'^2 = 0 \tag{28}$$

$$s'' + \mathrm{Le} f s' + \frac{N_t}{N_b}\theta'' = 0 \tag{29}$$

subject to the following boundary conditions:

$$\begin{aligned} &f' = 1, \quad f = 0, \quad \theta = 1, \quad s = 1 \quad \text{at} \quad \eta = 0 \\ &f' \to 0, \quad \theta \to 0, \quad s \to 0 \quad \text{as} \quad \eta \to \infty \end{aligned} \tag{30}$$

It is important to mentions that all the parameters appearing in the previous equations are free of independent variables unlike some previous studies on the convective flow of nanofluid. It is also important to mention that recently Xu and Pop [18] has obtained the ODE with constant parameters subject to non-zero free stream velocity.

The results and discussion

The physical quantities of practical interest *i. e.*, the local Nusselt number and the heat flux at the surface q_w are defined:

$$\text{Nu} = \frac{x q_w}{k\left(T_\text{w} - T_\infty\right)}, \quad q_w = -k\left(T_\text{w} - T_\infty\right) x^{-\frac{1}{4}} \frac{1}{2}\sqrt{\frac{3u_o}{\nu}}\theta'(0)$$

From this two expressions, conveniently we can write:

$$\text{Re}_x^{-1/2}\,\text{Nu} = -\theta'(0)$$

which is defined as reduced Nusselt number and will denote as Nu_r. The value of reduced Nusselt number is calculated for 75 sets of values of parameters N_r, N_b, and N_t and linear regression is performed on the data which yield the correlation:

$$\text{Nu}_\text{rest} = 0.46839 - 0.08901 N_r - 0.19428 N_b - 0.13118 N_t \tag{31}$$

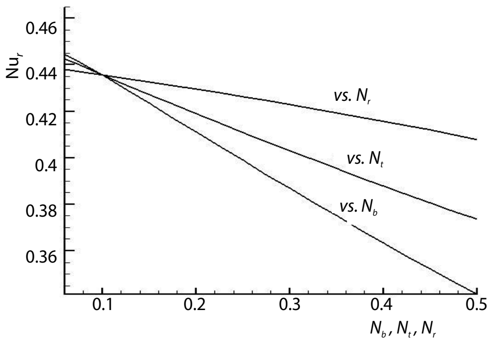

Figure 1. Reduce Nusselt number against Brownian motion parameter, thermophoresis parameter, and nano-particles concentration flux parameter for Pr = 0.5, Le = 1.0, and λ = 1.0. For each curve the other two parameters are fixed to be 0.1

The previous expression for estimated reduced Nusselt number is obtained for Pr = 0.5, Le = 1.0, and λ = 1.0. This can be seen from the previous expression that reduced Nusselt number decrease with an increase in each parameter N_r, N_b, and N_t.

The same behavior of reduced Nusselt number can be observed from fig. 1. For different values of Prandtl, Lewis, and Richardson numbers correlation for the reduced Nusselt number is given in tab. 1. To establish the reliability of correlation of estimated reduced Nusselt number the maximum percentage error is displayed in tab. 1. The percentage error in all the cases is less than 4% which confirm the reliability of these expressions for the practical purposes.

Table 1. Correlation of reduced Nusselt number ($\text{Nu}_r = C + C_{N_r}N_r + C_{N_b}N_b + C_{N_t}N_t$) and maximum percentage error [100 (Nu_Rest – Nu)/Nu] when the values of Brownian, thermophoresis and buoyancy ratio parameters are considered in the interval (0, 0.5)

Pr	Le	λ	C	C_{N_r}	C_{N_b}	C_{N_t}	Max. % error
0.5	1.0	1.0	0.468	–0.089	–0.194	0.131	2.9020
1.0	1.0	1.0	0.675	–0.104	–0.241	–0.169	3.1946
1.0	1.0	2.0	0.743	–0.105	–0.275	–0.181	3.8899
1.0	2.0	2.0	0.721	–0.046	–0.306	–0.201	3.6851
2.0	5.0	5.0	1.123	–0.019	–0.528	–0.309	2.60855

In similar fashion the estimated expression for the Sherwood number can be written. For illustration we are writing one expression for the particular values Pr = 0.5, Le = 1.0, and $\lambda = 1.0$.

$$\mathrm{Sh}_{Rest} = 0.395 - 0.443N_r + 1.102N_b - 0.586N_t \tag{32}$$

From previous correlation this can be seen that the reduce Sherwood number increase with an increase in N_b and decrease with an increase N_r and N_b. Expressions (31) and (32) show that rate of change with respect to N_r is very small. These observations can also be read from the numerical results which are shown in fig. 2.

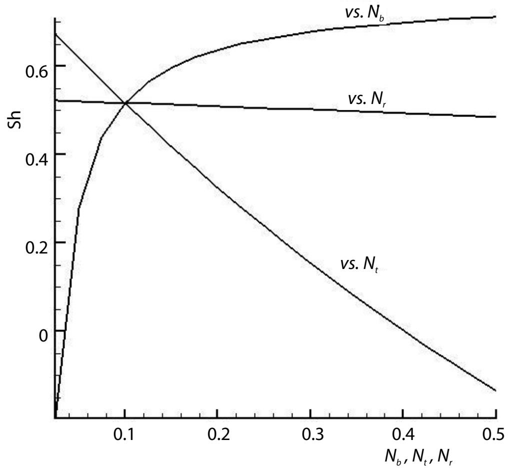

Figure 2. Reduce Sherwood number against Brownian motion parameter, thermophoresis parameter, and nano-particles concentration flux parameter for Pr = 0.5, Le =1.0, and λ = 1.0. For each curve the other two parameters are fixed to be 0.1

Conclusions

Mixed convection-boundary-layer flow of a viscous nanofluid past a vertical wall stretching with non-linear velocity is discussed. Similarity transformations are achieved using group theoretic method. The governing non-linear PDE are transformed into ODE using these transformations. It is shown that similarity solutions are possible only for the non-linear stretching velocity having form $x^{1/2}$. Reduced Nusselt and Sherwood numbers are given through correlation expression by performing linear regression on the numerical results. It is observed that both the reduced Nusselt number and Sherwood number decrease with the increase in nano-particles concentration flux, and Brownian motion parameters. However, increasing thermophoresis parameter reduced Nusselt number is decreased and Sherwood number is increased. Analytical expressions and numerical results ascertain that nano-particles concentration flux parameter plays comparatively role in the variation of reduced Nusselt number and Sherwood number as compared to other parameters.

Nomenclature

a, b, c, d, e, p, and *q* – scaling constants, [–]
C_1, C_2 – similarity constants, [–]
c – specific heat, [Jkg^{-1}K^{-1}]
D_B – Brownian diffusion coefficient, [m^{2}s^{-1}]
D_T – thermophoretic diffusion coefficient, [m^{2}s^{-1}]
f – dimensionless stream function, [–]
g – gravitational acceleration, [ms^{-2}]
L – reference length, [m]
Le – Lewis number, (= ν / D_B) [–]
N_b – Brownian motion parameter, (= $\tau D_B \Delta\phi / \nu$) [–]
N_r – nanoparticles concentration flux parameter, [= $L\Delta\phi(\rho_p - \rho_f)g / \rho_f U_0^2$], [–]
N_t – thermophoresis parameter, (= $\tau D_T \Delta T / \nu T_\infty$), [–]
Nu – local Nusselt number, [= $-\theta'(0)$] , [–]
Nu_r – reduced Nusselt number, (= $\mathrm{Re}_x^{-1/2}$ Nu), [–]
Pr – Prandtl number, (= ν / α) , [–]
q_w – surface heat flux, [–]
Re_x – local Reynolds number,(= U_x / ν) , [–]
s – dimensionless nanoparticle fraction, [= $(\phi - \phi_\infty) / (\phi_w - \phi_\infty)$], [–]
Sh – Sherwood number, [= $-s'(0)$] , [–]
T – fluid temperature, [K]
U – stretching velocity, [m s^{-1}]
u, v – velocity components along x- and y-axes, [m s^{-1}]
U_0 – reference velocity, [ms^{-1}]
x, y – Cartesian co-ordinates, [m]

Greek symbols

α – thermal diffusivity, (= κ / ρ_c), [m^{2}s^{-1}]

β – volumetric volume expansion coefficient, [K^{-1}]
ε – small perturbation parameter, [–]
η – similarity variable, [$= \bar{y} / \bar{x}^{1/4}$] [–]
λ – Richardson number, [$= L\Delta T(1 - \phi_\infty)g\beta / U_0^2$], [–]
μ – dynamic viscosity of the fluid, [$kgm^{-1}s^{-1}$]
ν – kinematic viscosity of the fluid, [m^2s^{-1}]
ϕ – nano-particle fraction, [–]
ρ – density, [kgm^{-3}]
ψ – stream function, [$=(2/3^{1/2})\bar{x}^{3/4}f(\eta)$], [$m^2s^{-1}$]
τ – ratio of effective heat capacity of the nanoparticle material to the heat capacity of the fluid, [$=(\rho c)_p / (\rho c)_f$], [–]
θ – dimensionless temperature, [$=(T - T_\infty) / (T_w - T_\infty)$], [–]

Superscripts

$'$ – differentiation with respect to η
$\bar{}$ – dimensionless quantity
$*$ – transformed co-ordinates under scaling group of transformation

Subscripts

f – base fluid
p – nano particle
w – condition at the wall
∞ – ambient condition

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