THE CONSERVATIVE DIFFERENCE SCHEME FOR THE GENERALIZED ROSENAU-KDV EQUATION

by

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In this paper, numerical solutions for the generalized Rosenau-KdV equation are considered via the energy and momentum conservative non-linear implicit finite difference scheme. Unique existence of the conservative properties of the solutions for the difference scheme is shown. Numerical results demonstrate that the scheme is efficient and reliable.

Key words: *generalized Rosenau-KdV equation, difference scheme, properties, numerical experiment*

Introduction

The well-known Korteweg-de Vries (KdV) equation [1-4]:

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

has been used to describe the wave propagation and spread interaction.

In this paper, we consider the following generalized Rosenau-KdV equation:

$$u_t + u_x + u_{xxx} + u_{xxxxt} + (u^p)_x = 0 \tag{2}$$

where $p \geq 2$ is an integer. When $p = 2$, eq. (2) is called as usual Rosenau-KdV equation.

In [1, 2], the solitary solutions for the generalized Rosenau-KdV equation with usual solitary ansatze method were discussed and the two invariants for the generalized Rosenau-KdV equation were given. In [2], the two types of soliton solution, *i. e.*, solitary wave solution and singular soliton, were researched. Furthermore, they also used perturbation theory and semi-variation principle to study the perturbed generalized Rosenau-KdV equation analytically. In [3], ansatze method was applied to obtain the topological soliton solution or shock solution of this equation. Moreover, three methods, that is, ansatze method, *G'/G*-expansion method as well as the exp-function method were applied to extract a few more solutions to this equation in [4]. But the numerical method to the initial-boundary value problem of generalized Rosenau-KdV equation has not been studied till now.

In [5, 6], two conservative difference schemes for the generalized Rosenau-KdV equation were proposed. But their schemes can only preserve one conservative law.

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In this paper, we propose a conservative non-linear Crank-Nicolson-implicit difference scheme for the equation. The studies show that the scheme does not need to select another scheme to help initial computation such as the average linear scheme in [5]. It should be noted that the numerical simulations show that the scheme preserves two conservative invariants, which is better than those results in [5, 6].

Hence, in this paper, we propose a conservative two-level non-linear implicit finite difference scheme for the generalized Rosenau-KdV eq. (2) with the boundary conditions:

$$u(X_l,t)=u(X_r,t)=0, \qquad u_x(X_l,t)=u_x(X_r,t)=0, \\ u_{xx}(X_l,t)=u_{xx}(X_r,t)=0, \qquad t\in[0,T] \tag{3}$$

and initial condition:

$$u(x,0)=u_0(x) \tag{4}$$

The initial-boundary value problem presents the following conservative properties [1, 2]:

$$M(t)=\int_{X_l}^{X_r} u\,\mathrm{d}x=\int_{X_l}^{X_r} u_0\,\mathrm{d}x=M(0) \tag{5}$$

$$E(t)=\int_{X_l}^{X_r}(u^2+u_{xx}^2)\mathrm{d}x=\|u\|_{L_2}^2+\|u_{xx}\|_{L_2}^2=E(0) \tag{6}$$

When $X_l \ll 0,\ X_r \gg 0,$ the initial-boundary value problem, eqs. (2)-(4), and the Cauchy problem, eq. (2), are consistent.

Conservative implicit difference scheme

In this section, we first give some notation used in this paper, and propose the conservative difference scheme for the problem of eqs. (2)-(4).

As usual, denote $x_j = X_l + jh,\ t_n = n\tau,\ 0 \le j \le J,\ 0 \le n \le N$, where $h = (X_r - X_l)/J$ and τ are the uniform the spatial and temporal step size, respectively. Let $u_j^n \approx u(jh,n\tau)$, $Z_h^0=\{u=(u_j)\big|u_{-1}=u_0=u_J=u_{J+1}=0,\ -1\le j\le J+1\}$.

Throughout this paper, we denote C as a general constant independent of h and τ.

Define the difference operators, inner product and norms are:

$$(u_j^n)_x=\frac{u_{j+1}^n-u_j^n}{h},\quad (u_j^n)_{\bar{x}}=\frac{u_j^n-u_{j-1}^n}{h},\quad (u_j^n)_{\hat{x}}=\frac{u_{j+1}^n-u_{j-1}^n}{2h},$$

$$(u_j^n)_t=\frac{u_j^{n+1}-u_j^n}{\tau},\quad (u_j^n)_{\hat{t}}=\frac{u_j^{n+1}-u_j^{n-1}}{2\tau},\quad (u_j^n)_{x\bar{x}}=\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2},$$

$$u_j^{n+\frac{1}{2}}=\frac{u_j^{n+1}+u_j^n}{2},\quad \left\langle u^n,v^n\right\rangle=h\sum_{j=1}^{J-1}u_j^n v_j^n,\quad \left\|u^n\right\|^2=\left\langle u^n,u^n\right\rangle,\quad \left\|u^n\right\|_\infty=\max_{0\le j\le J-1}\left|u_j^n\right|$$

In view of $(u^p)_x = 2/(1+p)\Sigma_{i=0}^{p-1}u^i(u^{p-i})_x$ [7], we can construct the following conservative implicit finite difference scheme for the problems, eqs. (2)-(4):

$$(u_j^n)_t+(u_j^{n+1/2})_{\hat{x}}+(u_j^{n+1/2})_{x\bar{x}\hat{x}}+(u_j^n)_{xx\bar{x}\bar{x}t}+\frac{2}{1+p}\sum_{i=0}^{p-1}\left(u_j^{n+1/2}\right)^i\left[\left(u_j^{n+1/2}\right)^{p-i}\right]_{\hat{x}}=0 \tag{7}$$

$$u_j^0 = u_0(x_j) \quad 1 \le j \le J-1 \tag{8}$$

$$u_0^n = u_J^n = 0\,,\ (u_0^n)_{\hat{x}} = (u_J^n)_{\hat{x}} = 0\,,\ (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0 \tag{9}$$

Lemma 2.1. [8] *For any two mesh functions* $u, v \in Z_h^0$, *one has:*

$$\langle v_x, u\rangle = -\langle v, u_{\bar{x}}\rangle, \quad \langle u_{\hat{x}}, v\rangle = -\langle u, v_{\hat{x}}\rangle, \quad \langle u, v_{x\bar{x}}\rangle = -\langle u_x, v_x\rangle \tag{10}$$

Then we have:

$$\langle u, u_{x\bar{x}}\rangle = -\langle u_x, u_x\rangle = -\|u_x\|^2 \tag{11}$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0$, then:

$$\langle u, u_{xx\bar{x}\bar{x}}\rangle = \|u_{xx}\|^2 \tag{12}$$

To prove the existence of solution for scheme, eqs. (7)-(9), the following Browder fixed point theorem should be introduced. For the proof, see [9].

Lemma 2.2. *(Browder fixed point theorem) Let H be a finite dimensional inner product space. Suppose that* $g: H \to H$ *is continuous and there exists an* $\alpha \ge 0$ *such that* $\langle g(x), x\rangle > 0,\ \forall x \in H,\ \|x\| = \alpha$. *Then there exists* $x^* \in H$ *such that* $g(x^*) = 0$ *and* $\|x^*\| \le \alpha$.

Theorem 2.3. *There exists* $u^n \in Z_h^0$ *satisfying the difference scheme (7)-(9).*

Proof: For $n \le N-1$, we assume that $u^0, u^1, \cdots, u^n \in Z_h^0$ satisfy the difference scheme (2.1)-(2.3). Next we prove that there exists u^{n+1} satisfying eqs. (7)-(9).

Define an operator g on Z_h^0 as follows:

$$g(v) = 2v - 2u^n + \tau v_{\hat{x}} + \tau v_{x\bar{x}\hat{x}} + 2v_{xx\bar{x}\bar{x}} - 2u^n_{xx\bar{x}\bar{x}} + \frac{2\tau}{1+p}\sum_{i=0}^{p-1}(v^{n+1})^i\left[(v^{n+1})^{p-i}\right]_{\hat{x}} = 0 \tag{13}$$

By computing the inner product of eq, (13) with v, we get:

$$\langle v_{\hat{x}}, v\rangle = 0\,, \quad \langle v_{x\bar{x}\hat{x}}, v\rangle = 0\,, \quad \left\langle \frac{2}{1+p}\sum_{i=0}^{p-1} v^i (v^{p-i})_{\hat{x}}, v\right\rangle = 0$$

Therefore:

$$\begin{aligned}
\langle g(v), v\rangle &= 2\|v\|^2 - 2\langle u^n, v\rangle + 2\|v_{xx}\|^2 - 2\langle u^n_{xx}, v_{xx}\rangle \ge \\
&\ge 2\|v\|^2 - 2\|u^n\|\cdot\|v\| + 2\|v_{xx}\|^2 - 2\|u^n_{xx}\|\cdot\|v_{xx}\| \ge \\
&\ge 2\|v\|^2 - \left(\|u^n\|^2 + \|v\|^2\right) + 2\|v_{xx}\|^2 - \left(\|u^n_{xx}\|^2 + \|v_{xx}\|^2\right) \ge \\
&\ge \|v\|^2 - \left(\|u^n\|^2 + \|u_{xx}\|^2\right) + \|v_{xx}\|^2 \ge \\
&\ge \|v\|^2 - \left(\|u^n\|^2 + \|u^n_{xx}\|^2\right)
\end{aligned} \tag{14}$$

It is obvious that $\langle g(v), v\rangle \ge 0$, for all $v \in Z_h^0$ with $\|v\|^2 = \|u^n\|^2 + \|u_{xx}\|^2 + 1$. It follows from Lemma 2.2 that there exists $v^* \in Z_h^0$ such that $g(v^*) = 0$. Let $u^{n+1} = 2v^* - u^n$, and it can be proved that u^{n+1} is the solution of the scheme (7)-(9).

Lemma 2.4. [6] *Suppose that* $u \in H_0^2[X_l, X_r]$, *then the solution of the initial-boundary value problem (2)-(4) satisfies:*

$$\|u\|_{L_2} \le C\,, \quad \|u_x\|_{L_2} \le C\,, \quad \|u\|_\infty \le C \tag{15}$$

Theorem 2.5. *Suppose that* $u_0 \in H_0^2[X_l, X_r]$, *then the schemes (7)-(9) are conservative for discrete momentum and energy, that is:*

$$M^n = h\sum_{j=1}^{J-1} u_j^n = M^{n-1} = \cdots = M^0 \tag{16}$$

$$E^n = \left\|u^n\right\|^2 + \left\|u_{xx}^n\right\|^2 = E^{n-1} = \cdots = E^0 \tag{17}$$

Proof: Multiplying eq. (7) with h and summing up for j from 1 to $J-1$, from the boundary condition in eq. (9), and lemma 2.1, we get:

$$h\sum_{j=1}^{J-1}(u_j^{n+1} - u_j^n) = 0 \tag{18}$$

Therefore, eq. (16) is easily gotten from eq. (18).

By computing the inner product of eq. (7) with $2u^{n+\frac{1}{2}}$ (*i. e.* $u^{n+1} + u^n$), we have:

$$h\sum_{j=1}^{J-1}\left\{\frac{1}{\tau}(u_j^{n+1} - u_j^n)2u_j^{n+\frac{1}{2}} + \left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}} 2u_j^{n+\frac{1}{2}} + \left(u_j^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}} 2u_j^{n+\frac{1}{2}} + \right.$$

$$\left. + \frac{1}{\tau}\left[(u_j^{n+1})_{xx\bar{x}\bar{x}} - (u_j^n)_{xx\bar{x}\bar{x}}\right]2u_j^{n+\frac{1}{2}} + 2P_j u_j^{n+\frac{1}{2}}\right\} = 0 \tag{19}$$

where

$$P_j = \frac{2}{p+1}\sum_{i=0}^{p-1}\left(u_j^{n+\frac{1}{2}}\right)^i\left[\left(u_j^{n+\frac{1}{2}}\right)^{p-i}\right]_{\hat{x}}$$

By the definition of $(u_j^n)_t$, it follows from the first term of eq. (19) that:

$$\sum_{j=1}^{J-1}\left(\frac{1}{\tau}(u_j^{n+1} - u_j^n)\cdot 2u_j^{n+\frac{1}{2}}\right) = \frac{1}{\tau}\left(\| u^{n+1} \|^2 - \| u^n \|^2\right) \tag{20}$$

From eq. (10), it follows from the second and the third term of eq. (19) that:

$$\sum_{j=1}^{J-1}\left[\left(u_j^{n+\frac{1}{2}}\right)_{\hat{x}} 2u_j^{n+\frac{1}{2}}\right] = \sum_{j=1}^{J-1}\left[\left(u_j^{n+\frac{1}{2}}\right)_x + \left(u_j^{n+\frac{1}{2}}\right)_{\bar{x}}\right]u_j^{n+\frac{1}{2}} =$$

$$= \sum_{j=1}^{J-1}\left(u_j^{n+\frac{1}{2}}\right)_x u_j^{n+\frac{1}{2}} + \sum_{j=1}^{J-1}\left(u_j^{n+\frac{1}{2}}\right)_{\bar{x}} u_j^{n+\frac{1}{2}} = 0 \tag{21}$$

Similarly:

$$\sum_{j=1}^{J-1}\left(u_j^{n+\frac{1}{2}}\right)_{x\bar{x}\hat{x}} 2u_j^{n+\frac{1}{2}}=0 \tag{22}$$

With the help of the boundary condition in eqs. (9) and (12), it follows from the fourth term of eq. (19) that:

$$\sum_{j=1}^{J-1}\frac{1}{\tau}[(u_j^{n+1})_{xx\bar{x}\bar{x}}-(u_j^n)_{xx\bar{x}\bar{x}}]2u_j^{n+\frac{1}{2}}=\frac{1}{\tau}\left(\left\|u_{xx}^{n+1}\right\|^2-\left\|u_{xx}^{n}\right\|^2\right) \tag{23}$$

In view of eq. (10), it follows from the last term of eq. (19) that:

$$\begin{aligned}\left\langle P,2u^{n+\frac{1}{2}}\right\rangle&=\frac{2h}{p+1}\sum_{j=1}^{J-1}\left\{\sum_{i=0}^{p-1}\left(u_j^{n+\frac{1}{2}}\right)^i\left[\left(u_j^{n+\frac{1}{2}}\right)^{p-i}\right]_{\hat{x}}\right\}u_j^{n+\frac{1}{2}}\\&=\frac{2h}{p+1}\sum_{j=1}^{J-1}\sum_{i=0}^{p-1}\left(u_j^{n+\frac{1}{2}}\right)^{i+1}\left[\left(u_j^{n+\frac{1}{2}}\right)^{p-\mathrm{i}}\right]_{\hat{x}}\\&=-\frac{2h}{p+1}\sum_{j=1}^{J-1}\sum_{i=0}^{p-1}\left(u_j^{n+\frac{1}{2}}\right)^{p-i-1}\left[\left(u_j^{n+\frac{1}{2}}\right)^{i+1}\right]_{\hat{x}}u_j^{n+\frac{1}{2}}\end{aligned} \tag{24}$$

Let $i'=p-(i+1)$. Obviously, if $i=0$, then $i'=p-1$. If $i=p-1$, then $i'=0$. It follows from eq. (24) that:

$$\left\langle P,2u^{n+\frac{1}{2}}\right\rangle=-\frac{2h}{p+1}\sum_{j=1}^{J-1}\sum_{i'=p-1}^{0}\left(u_j^{n+\frac{1}{2}}\right)^{i'}\left[\left(u_j^{n+\frac{1}{2}}\right)^{p-i'}\right]_{\hat{x}}u_j^{n+\frac{1}{2}}=-\left\langle P,2u^{n+\frac{1}{2}}\right\rangle \tag{25}$$

Therefore:

$$\left\langle P,2u^{n+\frac{1}{2}}\right\rangle=0 \tag{26}$$

By the previous results of eqs. (20)-(23) and eq. (26), we have:

$$\left(\left\|u^{n+1}\right\|^2-\left\|u^{n}\right\|^2\right)+\left(\left\|u_{xx}^{n+1}\right\|^2-\left\|u_{xx}^{n}\right\|^2\right)=0 \tag{27}$$

Then, by the definition of E^n, eq. (17) holds, which implies that the difference scheme is conservative for energy.

In order to prove the boundedness and the conservative law of the numerical solutions, we lead into the following lemma [8].

Lemma 2.6 (Discrete Sobolev's inequality) *There exist two constant* C_1 *and* C_2 *such that:*

$$\left\|u^n\right\|_\infty\le C_1\left\|u^n\right\|+C_2\left\|u_x^n\right\| \tag{28}$$

Theorem 2.7. *Suppose* $u_0 \in H_0^2[X_l, X_r]$, *then the solutions* u^n *of* eqs. (7)-(9) satisfy:

$$\|u^n\| \le C, \quad \|u_x^n\| \le C$$

which yields:

$$\|u^n\|_\infty \le C \ \ (n = 1, 2, \cdots, N)$$

Proof: It follows from eq. (17) that:

$$\|u^n\| \le C, \quad \|u_{xx}^n\| \le C \tag{29}$$

By Lemma 2.1 and Schwartz inequality, we get:

$$\|u_x^n\|^2 \le \|u^n\|\|u_{xx}^n\| \le \frac{1}{2}\left(\|u^n\|^2 + \|u_{xx}^n\|^2\right) \le C \tag{30}$$

From Lemma 2.6, we have $\|u^n\|_\infty \le C$ (n = 1, 2, …, N).

Numerical experiments

In this section, we present some numerical experiments to verify theoretical results obtained in previous sections.

We now consider two cases: p = 3 and p = 5, respectively.

When p = 3, the soliton solution is:

$$u(x,t) = \frac{1}{4}\sqrt{-15 + 3\sqrt{41}}\operatorname{sech}^2 \frac{1}{4}\sqrt{\frac{-5 + \sqrt{41}}{2}}\left[x - \frac{1}{10}(5 + \sqrt{41})t\right] \tag{31}$$

and the initial condition is:

$$u(x,0) = \frac{1}{4}\sqrt{-15 + 3\sqrt{41}}\operatorname{sech}^2 \frac{1}{4}\sqrt{\frac{-5 + \sqrt{41}}{2}}x \tag{32}$$

When p = 5, the soliton solution is:

$$u(x,t) = \sqrt[4]{\frac{4}{15}(-5 + \sqrt{34})}\operatorname{sech}\frac{1}{3}\sqrt{-5 + \sqrt{34}}\left[x - \frac{1}{10}(5 + \sqrt{34})t\right] \tag{33}$$

and the initial condition is:

$$u(x,0) = \sqrt[4]{\frac{4}{15}(-5 + \sqrt{34})} \operatorname{sech}\frac{1}{3}\sqrt{-5 + \sqrt{34}}x \tag{34}$$

First, we simulate the wave graph of the numerical solution of the implicit non-linear scheme eqs. (7)-(9). The comparison of numerical solution $u(x_j, t_n)$ between different time step and space step at various times are given in fig. 1 when p = 3. Similarly, we can get the almost same figures for different time step and space step at different times when p = 5, respectively.

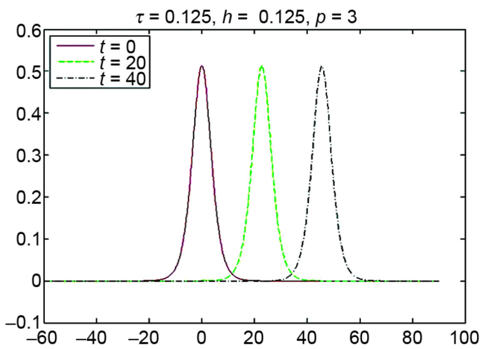

Figure 1. Wave graph of $u(x, t)$ at various times when p = 3 and $\tau = h$ = 0.125

Meanwhile, we also list the conservation invariants M^n and E^n at different time in tabs. 1 and 2 when p = 3. Similarly, we can get the conservation invariants M^n and E^n at different time when p = 5.

Table 1. The momentum of different time in different time step and space step when p = 3

(h, τ)	T = 10 s	T = 20 s	T = 30 s	T = 40 s
(1/4, 1/4)	4.898979485451472	4.898979484955972	4.898979615726935	4.898977554546477
(1/8, 1/8)	4.898979485364376	4.898979485171925	4.898979523325864	4.898978939201130
(1/16, 1/16)	4.898979485037968	4.898979485217347	4.898979495351347	4.898979312555465
(1/32, 1/32)	4.898979484150244	4.898979485429402	4.898979488895281	4.898979398515816

When p = 5. $M^n = 7.0936431010809$ and $E^n = 3.110702938788134$ at different time. These results also verify that the proposed scheme is conservative for two quantities M and E.

Table 2. The energy of different time in different time step and space step when p = 3

(h, τ)	T = 10 s	T = 20 s	T = 30 s	T = 40 s
(1/4, 1/4)	1.682528993311035	1.682528993310901	1.682528993310636	1.682528993309790
(1/8, 1/8)	1.682543082561832	1.682543082552275	1.682543082544440	1.682543082543480
(1/16, 1/16)	1.682546611165301	1.682546611191325	1.682546611097185	1.682546611292408
(1/32, 1/32)	1.682547493961059	1.682547495006936	1.682547495640218	1.682547495108805

Conclusion

In this paper, we constructed non-linear-implicit finite difference scheme for the general Rosenau-KdV equation and investigated some properties of its numerical solution. We proved that the non-linear scheme preserved the discrete mass and energy conservation, respectively. The proposed scheme is the unconditionally stable and second-order convergence by the discrete energy method. The results show that the scheme is reliable and efficient.

References

[1] Amin, E., Solitary Wave Solutions for Generalized Rosenau-KdV Equation, *Communications in Theoretical Physics*, *55* (2011), 3, pp. 396-398
[2] Polina, R., *et al.*, Perturbation of Dispersive Shallow Water Waves, *Ocean Engineering*, *63* (2013), May, pp. 1-7
[3] Saha, A., Topological 1-Soliton Solutions for the Generalized Rosenau-Kdv Equation, *Fundamental Journal of Mathematical Physics*, *2* (2012), 1, pp. 19-23
[4] Ghodrat, E., *et al.*, Topological Solitons and Other Solutions of the Rosenau-KdV Equation with Power Law Non-Linearity, *Romanian Journal of Physics*, *58* (2013), 1-2, pp. 1-10
[5] Zheng, M. B., *et al.*, An Average Linear Difference Scheme for the Generalized Rosenau-KdV Equation, *Journal of Applied Mathematics*, *2014* (2014), ID 202793
[6] Luo, Y., *et al.*, Conservative Difference Scheme for Generalized Rosenau-KdV Equation, *Advances in Mathematical Physics*, *2014* (2014), ID 986098
[7] Zuo, J. M., *et al.*, A New Conservative Difference Scheme for the General Rosenau-RLW Equation, *Boundary Value Problems*, *2010* (2010), ID 516260

[8] Zhou, Y., *Applications of Discrete Functional Analysis to the Finite Difference Method*, International Academic Publishers, Beijing, China, 1991
[9] Browder, F. E., Existence and Uniqueness Theorems for Solutions of Non-Linear Boundary Value Problems, *Proceedings,* Symposia in Applied Mathematics, New York, USA, 17 (1965), 6, pp. 24-49

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