THE LAPLACE SERIES SOLUTION FOR LOCAL FRACTIONAL KORTEWEG-DE VRIES EQUATION

by

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In this paper, we consider a new application of the local fractional Laplace series expansion method to handle the local fractional Korteweg-de Vries equation. The obtained solution with non-differentiable type shows that the technology is accurate and efficient.

Key words: *local fractional Korteweg-de Vries equation, analytical solution, Laplace series expansion method*

Introduction

Theory of local fractional calculus is a useful tool to model the non-differential problems in mathematical physics [1-3]. The local fractional partial differential equations (PDE) [4] are classified as one of three types, namely parabolic, hyperbolic, and elliptic. For example, the diffusion equation [5-9] is a parabolic local fractional PDE. The local fractional wave equation [10, 11] is hyperbolic local fractional PDE. The local fractional Laplace equation [12-14] is elliptic local fractional PDE.

Many technologies for solving the local fractional PDE were developed, *e. g.*, the functional method, Similarity variable method, differential transform method, Laplace variational iteration method, Laplace series expansion method, and so on. In this paper, we consider the local fractional Korteweg-de Vries equation [1]:

$$\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\tau^{\vartheta}}+\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{\vartheta}}+\frac{\partial^{3\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{3\vartheta}}=0 \tag{1}$$

where the local fractional partial derivative of $\Lambda(\psi,\tau)$ of order $\vartheta\,(0<\vartheta<1)$ at the point $\psi=\psi_0$ is given by [1, 4]:

$$\frac{\partial^{\vartheta}\Lambda(\psi_0,\tau)}{\partial\psi^{\vartheta}}=\frac{\Delta^{\vartheta}[\Lambda(\psi,\tau)-\Lambda(\psi_0,\tau)]}{(\psi-\psi_0)^{\vartheta}} \tag{2}$$

with $\Delta^{\vartheta}[\Lambda(\psi,\tau)-\Lambda(\psi_0,\tau)]\cong\Gamma(1+\vartheta)\Delta[\Lambda(\psi,\tau)-\Lambda(\psi_0,\tau)]$

The local fractional Laplace transform (LFLT) of $\Omega(\tau)$ is defined [1]:

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$$\widetilde{Y}_{\vartheta}\{\Omega(\tau)\} = \Omega_y^{\widetilde{Y},\vartheta}(y) = \frac{1}{\Gamma(1+\vartheta)}\int_0^{\infty} E_{\vartheta}(-y^{\vartheta}\tau^{\vartheta})\Omega(\tau)(\mathrm{d}\tau)^{\vartheta}, \quad 0<\vartheta\le 1 \tag{3}$$

The inverse LFLT is defined [1]:

$$\Omega(\tau) = \widetilde{Y}_{\vartheta}^{-1}\left\{\Omega_y^{\widetilde{Y},\vartheta}(y)\right\} = \frac{1}{(2\pi)^{\vartheta}}\int_{\mu-i\infty}^{\mu+i\infty} E_{\vartheta}(y^{\vartheta}\tau^{\vartheta})\Omega_y^{\widetilde{Y},\vartheta}(y)(\mathrm{d}y)^{\vartheta} \tag{4}$$

where $y^{\vartheta} = \mu^{\vartheta} + i^{\vartheta}\infty^{\vartheta}$ and $\mathrm{Re}(y^{\vartheta}) = \mu^{\vartheta}$. The properties of the LFLT were listed in [1].

The aim of the paper is to solve the local fractional Korteweg-de Vries equation by using the local fractional Laplace series expansion method.

The non-differentiable solution for local fractional Korteweg-de Vries equation

By using the technology [15], we present a multi-term separated function given by:

$$E_{\vartheta}(\tau^{\vartheta}) = \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)} \tag{5}$$

which leads to the solution of eq. (1), namely:

$$\Lambda(\psi,\tau) = \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_i(\psi) \tag{6}$$

where $\mathrm{H}_i(\psi)$ is the non-differentiable functions with respect to ψ.

With the help of eq. (6), we have:

$$\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\tau^{\vartheta}} = \frac{\partial^{\vartheta}}{\partial\tau^{\vartheta}}\left[\sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_k(\psi)\right] = \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_{k+1}(\psi) \tag{7}$$

$$\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{\vartheta}} = \frac{\partial^{\vartheta}}{\partial\psi^{\vartheta}}\left[\sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_k(\psi)\right] = \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\frac{\partial^{\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{\vartheta}} \tag{8}$$

and

$$\frac{\partial^{3\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{3\vartheta}} = \frac{\partial^{3\vartheta}}{\partial\psi^{3\vartheta}}\left[\sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_k(\psi)\right] = \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\frac{\partial^{3\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{3\vartheta}} \tag{9}$$

After substituting eqs. (7)-(9) into eq. (1), we have:

$$\sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\mathrm{H}_{k+1}(\psi) + \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\frac{\partial^{\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{\vartheta}} + \sum_{k=0}^{\infty}\frac{\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\frac{\partial^{3\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{3\vartheta}} = 0 \tag{10}$$

which leads to the operator equation:

$$\mathrm{H}_{k+1}(\psi) = -\left[\frac{\partial^{\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{\vartheta}} + \frac{\partial^{3\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{3\vartheta}}\right] \tag{11}$$

We now consider the initial value condition of eq. (1), namely:

$$\Lambda(\psi,0)=E_{\vartheta}(-\psi^{\vartheta}) \tag{12}$$

Thus, we have the following iterative formula:

$$\begin{cases} \mathrm{H}_{k+1}(\psi)=-\left[\dfrac{\partial^{\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{\vartheta}}+\dfrac{\partial^{3\vartheta}\mathrm{H}_k(\psi)}{\partial\psi^{3\vartheta}}\right] \\ \mathrm{H}_0(\psi)=E_{\vartheta}(-\psi^{\vartheta}) \end{cases} \tag{13}$$

Taking the LFLT of eq. (13), we present a new iteration formula:

$$\begin{cases} \mathrm{H}_{k+1}(y)=(-2)^k\mathrm{H}_k(y) \\ \mathrm{H}_0(y)=\dfrac{1}{1+y^{\vartheta}} \end{cases} \tag{14}$$

Therefore, we obtain from eq. (13) the following values:

$$\mathrm{H}_1(y)=\frac{-2}{1+y^{\vartheta}} \tag{15}$$

$$\mathrm{H}_2(\psi)=\frac{2^2}{1+y^{\vartheta}} \tag{16}$$

$$\mathrm{H}_k(\psi)=\frac{-2^k}{1+y^{\vartheta}} \tag{17}$$

and others.

Thus, we obtain the local fractional Laplace series of the solution of eq. (1), namely:

$$\Lambda(y,\tau)=\sum_{k=0}^{\infty}\frac{-2^k\tau^{k\vartheta}}{\Gamma(1+k\vartheta)}\frac{1}{1+y^{\vartheta}}=E_{\vartheta}(-2\tau^{\vartheta})\frac{1}{1+y^{\vartheta}} \tag{18}$$

Taking the inverse LFLT of eq. (18), we give the local fractional series solution:

$$\Lambda(\psi,\tau)=E_{\vartheta}(-2\tau^{\vartheta})E_{\vartheta}(\psi^{\vartheta}) \tag{19}$$

and its corresponding graph represents in fig. 1.

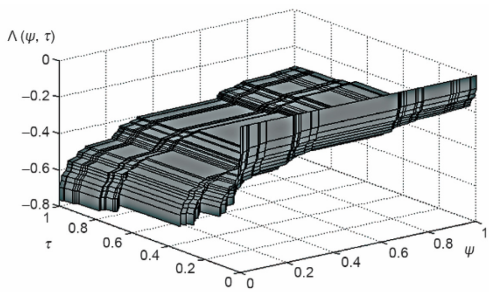

Figure 1. The series solution of local fractional Korteweg-de Vries equation when ϑ = ln2/ln3

Conclusion

We use the local fractional Laplace series expansion method to deal with the local fractional Korteweg-de Vries equation. The non-differentiable solution when ϑ = ln2/ln3 is obtained. The new technology is accurate and efficient for solving the PDE in mathematical physics.

Nomenclature

ψ – space co-ordinate, [m]
ϑ – fractal dimensional order, [–]
$\widetilde{Y}_{\vartheta}\{\Omega(\tau)\}$ – LFLT of $\Omega(\tau)$, [–]
τ – time, [s]
$\widetilde{Y}_{\vartheta}^{-1}\{\Omega_y^{\widetilde{Y},\vartheta}(y)\}$ – inverse LFLT, [–]

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