ABOUT LOCAL FRACTIONAL THREE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS IN CANTOR-TYPE CYLINDRICAL CO-ORDINATE SYSTEM

by

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In this article, we investigate the local fractional 3-D compressible Navier-Stokes equation via local fractional derivative. We use the Cantor-type cylindrical co-ordinate method to transfer 3-D compressible Navier-Stokes equation from the Cantorian co-ordinate system to the Cantor-type cylindrical co-ordinate system.

Key words: *compressible Navier-Stokes equation, local fractional derivative, Cantor-type cylindrical co-ordinate method*

Introduction

Local fractional partial differential equations were observed in different co-ordinate systems, such as heat-conduction equation [1], Helmholtz equation [2], Maxwell's equation [3], wave equation [4], and diffusion equation [5]. The Cantor-type spherical co-ordinate [1], Cantor-type circle co-ordinate [2], and Cantor-type cylindrical co-ordinate [6] methods were proposed and developed to describe the heat transfer problems. The 3-D compressible Navier-Stokes equation in the Cantorian co-ordinate system via local fractional derivative was reported in [2, 7, 8]. In this manuscript, we use the Cantor-type cylindrical co-ordinate method to transfer 3-D compressible Navier-Stokes equation from the Cantorian co-ordinate system to the Cantor-type cylindrical co-ordinate system.

The Cantor-type cylindrical co-ordinate method

In this section, we introduce the concept of the local fractional derivative and the Cantor-type cylindrical co-ordinate method.

The local fractional partial derivative of the function $\Phi_\kappa(\theta,\vartheta)$ of order $\kappa(0<\kappa<1)$ at $\theta=\theta_0$ is defined by [1, 2]:

$$\mathrm{D}_\theta^{(\kappa)}\Phi_\kappa(\theta_0,\vartheta)=\frac{\mathrm{d}^\kappa\Phi_\kappa(\theta,\vartheta)}{\mathrm{d}\theta^\kappa}\Big|_{\theta=\theta_0}=\lim_{\theta\to\theta_0}\frac{\Delta^\kappa[\Phi_\kappa(\theta,\vartheta)-\Phi_\kappa(\theta_0,\vartheta)]}{(\theta-\theta_0)^\kappa} \tag{1a}$$

where $\Delta^\kappa[\Phi_\kappa(\theta,\vartheta)-\Phi_\kappa(\theta_0,\vartheta)]\cong\Gamma(1+\kappa)\Delta[\Phi_\kappa(\theta,\vartheta)-\Phi_\kappa(\theta_0,\vartheta)]$.

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The Cantor-type cylindrical co-ordinates can be written [2, 4]:

$$\begin{cases} \mu^\kappa = R^\kappa \cos_\kappa(\theta^\kappa), \\ \eta^\kappa = R^\kappa \sin_\kappa(\theta^\kappa), \\ \sigma^\kappa = \sigma^\kappa \end{cases} \tag{1b}$$

where $R \in (0,+\infty), \sigma \in (-\infty,+\infty), \theta \in (0,2\pi], \mu^{2\kappa} + \eta^{2\kappa} = R^{2\kappa}$.

By using eq. (1), we have [2, 4]:

$$\nabla^\kappa \vec{\mathrm{r}} = \frac{\partial^\kappa r_R}{\partial R^\kappa} + \frac{1}{R^\kappa}\frac{\partial^\kappa r_\theta}{\partial \theta^\kappa} + \frac{r_R}{R^\kappa} + \frac{\partial^\kappa r_z}{\partial \sigma^\kappa} \tag{2a}$$

and

$$\nabla^\kappa \times \vec{\mathrm{r}} = \left(\frac{1}{R^\kappa}\frac{\partial^\kappa r_\theta}{\partial \theta^\kappa} - \frac{\partial^\kappa r_\theta}{\partial \sigma^\kappa}\right)\vec{\mathrm{e}}_R^\kappa + \left(\frac{\partial^\kappa r_R}{\partial \sigma^\kappa} - \frac{\partial^\kappa r_z}{\partial R^\kappa}\right)\vec{\mathrm{e}}_\theta^\kappa + \left(\frac{\partial^\kappa r_\theta}{\partial R^\kappa} + \frac{r_R}{R^\kappa} - \frac{1}{R^\kappa}\frac{\partial^\kappa r_R}{\partial \theta^\kappa}\right)\vec{\mathrm{e}}_\sigma^\kappa \tag{2b}$$

where

$$r = \mathrm{R}^\kappa \cos_\kappa(\theta^\kappa)\vec{\mathrm{e}}_1^\kappa + \mathrm{R}^\kappa \sin_\kappa(\theta^\kappa)\vec{\mathrm{e}}_2^\kappa + \sigma^\kappa \vec{\mathrm{e}}_3^\kappa = r_R \vec{\mathrm{e}}_R^\kappa + r_\theta \vec{\mathrm{e}}_\theta^\kappa + r_\sigma \vec{\mathrm{e}}_\sigma^\kappa \tag{2c}$$

The local fractional gradient and Laplace operators in the Cantor-type cylindrical co-ordinate system are written as [2, 4]:

$$\nabla^\kappa \phi(R,\theta,\sigma) = \vec{\mathrm{e}}_R^\kappa \frac{\partial^\kappa \phi}{\partial R^\kappa} + \vec{\mathrm{e}}_\theta^\kappa \frac{1}{R^\kappa}\frac{\partial^\kappa \phi}{\partial \theta^\kappa} + \vec{\mathrm{e}}_\sigma^\kappa \frac{\partial^\kappa \phi}{\partial \sigma^\kappa} \tag{3a}$$

$$\nabla^{2\kappa} \phi(R,\theta,\sigma) = \frac{\partial^{2\kappa}\phi}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\phi}{\partial \theta^{2\kappa}} + \frac{1}{R^\kappa}\frac{\partial^\kappa \phi}{\partial R^\kappa} + \frac{\partial^{2\kappa}\phi}{\partial \sigma^{2\kappa}} \tag{3b}$$

where a local fractional vector is [2, 4]:

$$\begin{cases} \vec{\mathrm{e}}_R^\kappa = \cos_\kappa(\theta^\kappa)\vec{\mathrm{e}}_1^\kappa + \sin_\kappa(\theta^\kappa)\vec{\mathrm{e}}_2^\kappa, \\ \vec{\mathrm{e}}_\theta^\kappa = -\sin_\kappa(\theta^\kappa)\vec{\mathrm{e}}_1^\kappa + \cos_\kappa(\theta^\kappa)\vec{\mathrm{e}}_2^\kappa, \\ \vec{\mathrm{e}}_\sigma^\kappa = \vec{\mathrm{e}}_3^\kappa \end{cases} \tag{3c}$$

The local fractional operator is written:

$$\nabla^\kappa(\nabla^\kappa \vec{\mathrm{A}}) = \frac{\partial^{2\kappa}\vec{\mathrm{A}}_R}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\vec{\mathrm{A}}_\theta}{\partial \theta^{2\kappa}} + \frac{\vec{\mathrm{A}}_R}{R^{2\kappa}} + \frac{\partial^{2\kappa}\vec{\mathrm{A}}_z}{\partial \sigma^{2\kappa}} + \frac{1}{\Gamma(1-2\kappa)R^{2\kappa}}\frac{\partial^\kappa \vec{\mathrm{A}}_\theta}{\partial \theta^\kappa} + \frac{\vec{\mathrm{A}}_R}{\Gamma(1-2\kappa)R^{2\kappa}} \tag{3d}$$

Transferring the 3-D compressible Navier-Stokes equation

For compressible fluid, the 3-D Navier-Stokes equation on Cantor sets without the specific fractal body force is written in the form:

$$\rho\frac{\partial^\kappa \vec{\upsilon}}{\partial t^\kappa} = -\nabla^\kappa p + \frac{\mu}{3}\nabla^\kappa[(\nabla^\kappa \vec{\upsilon})] + \mu\nabla^{2\kappa}\vec{\upsilon} - \rho\vec{\upsilon}(\nabla^\kappa \vec{\upsilon}) \tag{4a}$$

which reduces to:

$$\frac{\partial^{\kappa}\vec{\upsilon}}{\partial t^{\kappa}} = -\nabla^{\kappa} p + \frac{\varpi}{3}\nabla^{\kappa}[(\nabla^{\kappa}\vec{\upsilon})] + \varpi\nabla^{2\kappa}\vec{\upsilon} - \vec{\upsilon}(\nabla^{\kappa}\vec{\upsilon}) \tag{4b}$$

where p is the fractal pressure field, ρ – the fractal mass density, $\vec{\upsilon}$ – the fractal flow velocity, and μ – the dynamic viscosity.

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantorian co-ordinate system using eq. (4b) are written:

$$\frac{\partial^{\kappa} v_x}{\partial t^{\kappa}} = -\frac{\partial^{\kappa} P}{\partial x^{\kappa}} + \varpi\left(\frac{\partial^{2\kappa} v_x}{\partial x^{2\kappa}} + \frac{\partial^{2\kappa} v_x}{\partial y^{2\kappa}} + \frac{\partial^{2\kappa} v_x}{\partial z^{2\kappa}}\right) + \frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial x^{\kappa}}\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) - \\ - v_x\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) \tag{4c}$$

$$\frac{\partial^{\kappa} v_y}{\partial t^{\kappa}} = -\frac{\partial^{\kappa} P}{\partial y^{\kappa}} + \varpi\left(\frac{\partial^{2\kappa} v_y}{\partial x^{2\kappa}} + \frac{\partial^{2\kappa} v_y}{\partial y^{2\kappa}} + \frac{\partial^{2\kappa} v_y}{\partial z^{2\kappa}}\right) + \frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial y^{\kappa}}\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) - \\ - v_y\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) \tag{4d}$$

$$\frac{\partial^{\kappa} v_z}{\partial t^{\kappa}} = -\frac{\partial^{\kappa} P}{\partial z^{\kappa}} + \varpi\left(\frac{\partial^{2\kappa} v_z}{\partial x^{2\kappa}} + \frac{\partial^{2\kappa} v_z}{\partial y^{2\kappa}} + \frac{\partial^{2\kappa} v_z}{\partial z^{2\kappa}}\right) + \frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial z^{\kappa}}\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) - \\ - v_z\left(\frac{\partial^{\kappa} v_x}{\partial x^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial y^{\kappa}} + \frac{\partial^{\kappa} v_x}{\partial z^{\kappa}}\right) \tag{4e}$$

where v_x, v_y, and v_z, are the components of the fractal flow velocity v in the directions x, y, and z, respectively.

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantor-type cylindrical co-ordinate system are written:

$$\frac{\partial^{\kappa}\vec{\upsilon}}{\partial t^{\kappa}} = -\left(\vec{e}_R^{\kappa}\frac{\partial^{\kappa} P}{\partial R^{\kappa}} + \vec{e}_{\theta}^{\kappa}\frac{1}{R^{\kappa}}\frac{\partial^{\kappa} P}{\partial \theta^{\kappa}} + \vec{e}_{\sigma}^{\kappa}\frac{\partial^{\kappa} P}{\partial \sigma^{\kappa}}\right) + \varpi\left(\frac{\partial^{2\kappa}\vec{\upsilon}}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\vec{\upsilon}}{\partial \theta^{2\kappa}} + \frac{1}{R^{\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}}{\partial R^{\kappa}} + \frac{\partial^{2\kappa}\vec{\upsilon}}{\partial \sigma^{2\kappa}}\right) + \\ + \frac{\varpi}{3}\left(\frac{\partial^{2\kappa}\vec{\upsilon}_R}{\partial R^{2\kappa}} + \frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\vec{\upsilon}_{\theta}}{\partial \theta^{2\kappa}} + \frac{\vec{\upsilon}_R}{R^{2\kappa}} + \frac{\partial^{2\kappa}\vec{\upsilon}_z}{\partial \sigma^{2\kappa}} + \frac{1}{\Gamma(1-2\kappa)R^{2\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}_{\theta}}{\partial \theta^{\kappa}} + \frac{\vec{\upsilon}_R}{\Gamma(1-2\kappa)R^{2\kappa}}\right) - \\ - \vec{\upsilon}\left(\frac{\partial^{\kappa}\vec{\upsilon}_R}{\partial R^{\kappa}} + \frac{1}{R^{\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}_{\theta}}{\partial \theta^{\kappa}} + \frac{\vec{\upsilon}_R}{R^{\kappa}} + \frac{\partial^{\kappa}\vec{\upsilon}_z}{\partial \sigma^{\kappa}}\right) \tag{4f}$$

where $\vec{\upsilon}_R, \vec{\upsilon}_{\theta}$, and $\vec{\upsilon}_z$ are the components of the fractal flow velocity $\vec{\upsilon}$ in the Cantor-type cylindrical co-ordinate system, respectively.

For compressible fluid, the Navier-Stokes equation on Cantor sets with the specific fractal body force, $\vec{b}$, is written in the form:

$$\rho\frac{\partial^{\kappa}\vec{\upsilon}}{\partial t^{\kappa}}=-\nabla^{\kappa}p+\frac{\mu}{3}\nabla^{\kappa}[(\nabla^{\kappa}\vec{\upsilon})]+\mu\nabla^{2\kappa}\vec{\upsilon}+\rho\vec{\mathrm{b}}-\rho\vec{\upsilon}(\nabla^{\kappa}\vec{\upsilon}), \tag{5a}$$

which leads to:

$$\frac{\partial^{\kappa}\vec{\upsilon}}{\partial t^{\kappa}}=-\nabla^{\kappa}P+\frac{\varpi}{3}\nabla^{\kappa}[(\nabla^{\kappa}\vec{\upsilon})]+\varpi\nabla^{2\kappa}\vec{\upsilon}-\vec{\upsilon}(\nabla^{\kappa}\vec{\upsilon})+\vec{\mathrm{b}} \tag{5b}$$

where $\vec{\mathrm{b}}$ is the body acceleration, and $P = p/\rho$ and $\varpi = \mu/\rho$.

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantorian co-ordinate system are written as:

$$\frac{\partial^{\kappa}v_x}{\partial t^{\kappa}}=-\frac{\partial^{\kappa}P}{\partial x^{\kappa}}+\varpi\left(\frac{\partial^{2\kappa}v_x}{\partial x^{2\kappa}}+\frac{\partial^{2\kappa}v_x}{\partial y^{2\kappa}}+\frac{\partial^{2\kappa}v_x}{\partial z^{2\kappa}}\right)+$$
$$+\frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial x^{\kappa}}\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)-v_x\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)+b_x \tag{5c}$$

$$\frac{\partial^{\kappa}v_y}{\partial t^{\kappa}}=-\frac{\partial^{\kappa}P}{\partial y^{\kappa}}+\varpi\left(\frac{\partial^{2\kappa}v_y}{\partial x^{2\kappa}}+\frac{\partial^{2\kappa}v_y}{\partial y^{2\kappa}}+\frac{\partial^{2\kappa}v_y}{\partial z^{2\kappa}}\right)+$$
$$+\frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial y^{\kappa}}\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)-v_y\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)+b_y \tag{5d}$$

$$\frac{\partial^{\kappa}v_z}{\partial t^{\kappa}}=-\frac{\partial^{\kappa}P}{\partial z^{\kappa}}+\varpi\left(\frac{\partial^{2\kappa}v_z}{\partial x^{2\kappa}}+\frac{\partial^{2\kappa}v_z}{\partial y^{2\kappa}}+\frac{\partial^{2\kappa}v_z}{\partial z^{2\kappa}}\right)+$$
$$+\frac{\varpi}{3}\frac{\partial^{\kappa}}{\partial z^{\kappa}}\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)-v_z\left(\frac{\partial^{\kappa}v_x}{\partial x^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial y^{\kappa}}+\frac{\partial^{\kappa}v_x}{\partial z^{\kappa}}\right)+b_z \tag{5e}$$

The 3-D compressible Navier-Stokes equations on Cantor sets without the specific fractal body force in the Cantor-type cylindrical co-ordinate system are written as:

$$\frac{\partial^{\kappa}\vec{\upsilon}}{\partial t^{\kappa}}=-\left(\vec{\mathrm{e}}_R^{\kappa}\frac{\partial^{\kappa}P}{\partial R^{\kappa}}+\vec{\mathrm{e}}_{\theta}^{\kappa}\frac{1}{R^{\kappa}}\frac{\partial^{\kappa}P}{\partial\theta^{\kappa}}+\vec{\mathrm{e}}_{\sigma}^{\kappa}\frac{\partial^{\kappa}P}{\partial\sigma^{\kappa}}\right)+$$
$$+\frac{\varpi}{3}\left(\frac{\partial^{2\kappa}\vec{\upsilon}_R}{\partial R^{2\kappa}}+\frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\vec{\upsilon}_{\theta}}{\partial\theta^{2\kappa}}+\frac{\vec{\upsilon}_R}{R^{2\kappa}}+\frac{\partial^{2\kappa}\vec{\upsilon}_z}{\partial\sigma^{2\kappa}}+\frac{1}{\Gamma(1-2\kappa)R^{2\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}_{\theta}}{\partial\theta^{\kappa}}+\frac{\vec{\upsilon}_R}{\Gamma(1-2\kappa)R^{2\kappa}}\right)+$$
$$+\varpi\left(\frac{\partial^{2\kappa}\vec{\upsilon}}{\partial R^{2\kappa}}+\frac{1}{R^{2\kappa}}\frac{\partial^{2\kappa}\vec{\upsilon}}{\partial\theta^{2\kappa}}+\frac{1}{R^{\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}}{\partial R^{\kappa}}+\frac{\partial^{2\kappa}\vec{\upsilon}}{\partial\sigma^{2\kappa}}\right)-\vec{\upsilon}\left(\frac{\partial^{\kappa}\vec{\upsilon}_R}{\partial R^{\kappa}}+\frac{1}{R^{\kappa}}\frac{\partial^{\kappa}\vec{\upsilon}_{\theta}}{\partial\theta^{\kappa}}+\frac{\vec{\upsilon}_R}{R^{\kappa}}+\frac{\partial^{\kappa}\vec{\upsilon}_z}{\partial\sigma^{\kappa}}\right)+\vec{\mathrm{b}} \tag{5f}$$

Conclusion

In this work, we investigated the 3-D compressible Navier-Stokes equations on Cantor sets with and without the specific fractal body forces in the Cantorian co-ordinate systems. We applied the Cantor-type cylindrical co-ordinate method to transfer the 3-D compressible Navier-Stokes equations on Cantor sets in the Cantorian co-ordinate system into the Cantor--type cylindrical co-ordinate system.

Nomenclature

x, y, z – space co-ordinate, [m]
p – fractal pressure field, [$Pa \cdot m^{-3}$]

Greek symbols

κ – fractal dimension, [-]
ρ – fractal mass density, [kgm^{-3}]

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