# FRACTAL COMPLEX TRANSFORM TECHNOLOGY FOR FRACTAL KORTEWEG-DE VRIES EQUATION WITHIN A LOCAL FRACTIONAL DERIVATIVE

by

***Jinze XU[a], Zeng-Shun CHEN[b*], Jian-Hong WANG[c], Ping CUI[d], and Yunru BAI[e]***

[a] Department of Chemical and Petroleum Engineering, University of Calgary, Calgary, Canada
[b] State Key Laboratory Breeding Base of Mountain Bridge and Tunnel Engineering, and Key Laboratory of Bridge Structure Engineering Transportation Industry, Chongqing Jiaotong University, Chongqing, China
[c] School of Science, Nantong University, Nantong, China
[d] School of Teacher Education, Qujing Normal University, Qujing, China
[e] College of Sciences, Guangxi University for Nationalities, Nanning, China

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*In this paper, we present the fractal complex transform via a local fractional derivative. The traveling wave solutions for the fractal Korteweg-de Vries equations within local fractional derivative are obtained based on the special functions defined on Cantor sets. The technology is a powerful tool for solving the local fractional non-linear partial differential equations.*

Key words: *Korteweg-de Vries equation, traveling wave solution, fractal complex transform, local fractional derivative*

## Introduction

Fractional complex transform method is a powerful tool to solve the partial differential equations. The idea was extended to handle the partial differential equations in different operator. For example, He and Li [1] used it to covert a class of the fractional differential equations into partial differential equations. Yang *et al.* [2] considered that it transfers the classical differential equations into a class of the local fractional differential equations. Local fractional differential equations may be modeled to mathematical problems involving fractal engineering practice [3-6]. Recently, a new fractional derivative was proposed by Yang *et al.* [7] to describe the fractal relaxation and diffusion phenomena. We now recall the local fractional derivative.

The local fractional derivative of the function $\Omega(\eta)$ of order $\zeta(0<\zeta<1)$ is defined by [7]:

$$\mathrm{D}^{(\zeta)}\Omega(\eta)=\frac{\mathrm{d}^{\zeta}\Omega(\eta)}{\mathrm{d}\eta^{\zeta}}\Big|_{\eta=\eta_0}=\lim_{\eta\to\eta_0}\frac{\Omega(\eta)-\Omega(\eta_0)}{(\eta-\eta_0)^{\zeta}} \tag{1}$$

* Corresponding author; e-mail: chenzs2007@163.com

where $(\eta - \eta_0)^{\varepsilon}$ represents a fractal measure.

Based on the property of eq. (1), a fractal complex transform technology was proposed [7]. Our aim of the paper is to give new applications to the fractal Korteweg-de Vries equations.

## Special functions defined on Cantor sets

A sup-exponential functions defined on Cantor sets is defined by [7]:

$$\sup \exp(\eta^{\zeta}) = \sum_{i=0}^{\infty} \frac{\eta^{\zeta i}}{\Gamma(i+1)} \tag{2}$$

where $\zeta$ represents the fractal dimension.

Based on the exponential functions defined on Cantor sets, we define the special functions defined on Cantor sets:

$$\sup \exp(i^{\zeta} \eta^{\zeta}) = \sup \cos(\eta^{\zeta}) + i^{\zeta} \sup \sin(\eta^{\zeta}) \tag{3}$$

$$\sup \cos(\eta^{\zeta}) = \sum_{i=0}^{\infty} \frac{(-1)^i \eta^{2i\zeta}}{\Gamma(2i+1)} \tag{4}$$

$$\sup \sin(\eta^{\zeta}) = \sum_{i=0}^{\infty} \frac{(-1)^i \eta^{(2i+1)\zeta}}{\Gamma(2i+2)} \tag{5}$$

$$\sup \sinh(\eta^{\zeta}) = \frac{\sup \exp(\eta^{\zeta}) - \sup \exp(-\eta^{\zeta})}{2} \tag{6}$$

$$\sup \cosh(\eta^{\zeta}) = \frac{\sup \exp(\eta^{\zeta}) + \sup \exp(-\eta^{\zeta})}{2} \tag{7}$$

$$\sup \tanh(\eta^{\zeta}) = \frac{\sup \exp(\eta^{\zeta}) - \sup \exp(-\eta^{\zeta})}{\sup \exp(\eta^{\zeta}) + \sup \exp(-\eta^{\zeta})} \tag{8}$$

$$\sup \coth(\eta^{\zeta}) = \frac{\sup \exp(\eta^{\zeta}) + \sup \exp(-\eta^{\zeta})}{\sup \exp(\eta^{\zeta}) - \sup \exp(-\eta^{\zeta})} \tag{9}$$

$$\sup \operatorname{sech}(\eta^{\zeta}) = \frac{2}{\sup \exp(\eta^{\zeta}) + \sup \exp(-\eta^{\zeta})} \tag{10}$$

$$\sup \operatorname{csch}(\eta^{\zeta}) = \frac{2}{\sup \exp(\eta^{\zeta}) - \sup \exp(-\eta^{\zeta})} \tag{11}$$

where $i^{\zeta}$ is a imaginary on Cantor sets [3].

The graphs of the special functions defined on Cantor sets are presented in fig. 1.

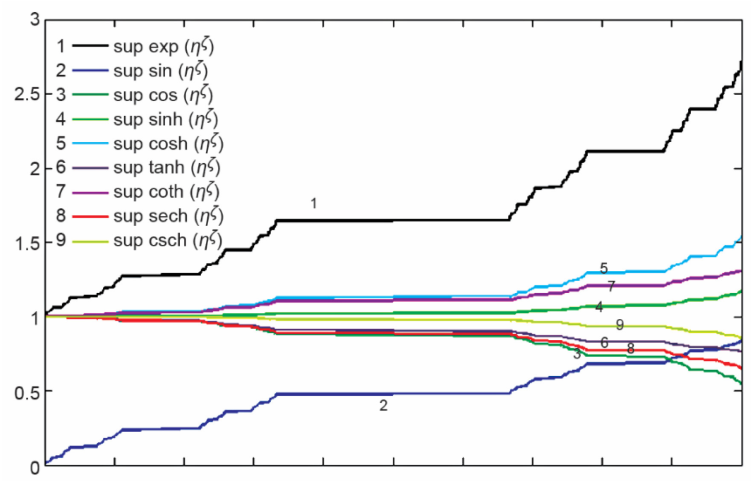


**Figure 1. The special functions for $\zeta$ = ln2/ln3**

## Solving fractal Korteweg-de Vries equations

We firstly consider the time-fractal Korteweg-de Vries equation:

$$\frac{\partial^{\zeta}\Theta(\eta,\tau)}{\partial\tau^{\zeta}}+6\Theta(\eta,\tau)\frac{\partial\Theta(\eta,\tau)}{\partial\eta}+\frac{\partial^{3}\Theta(\eta,\tau)}{\partial\eta^{3}}=0 \tag{12}$$

With the help of the fractal complex transform, which is given by:

$$\frac{\partial^{\zeta}\Theta[\eta,\mu(\tau)]}{\partial\tau^{\zeta}}=\frac{\partial\Theta[\eta,\mu(\tau)]}{\partial\mu}D^{(\zeta)}\tau^{\zeta} \tag{13}$$

where $\mu(\tau)=\tau^{\zeta}$, we get the classical Korteweg-de Vries equation:

$$\frac{\partial\Theta(\eta,\mu)}{\partial\mu}+6\Theta(\eta,\mu)\frac{\partial\Theta(\eta,\mu)}{\partial\eta}+\frac{\partial^{3}\Theta(\eta,\mu)}{\partial\eta^{3}}=0 \tag{14}$$

The solution of eq. (14) can be written in the form [8]:

$$\Theta(\eta,\mu)=-\frac{c}{2}\operatorname{sech}^{2}\left[\frac{\sqrt{c}}{2}(\eta-c\mu)\right] \tag{15}$$

Thus, we have the non-differentiable solution, namely:

$$\Theta(\eta,\tau)=-\frac{c}{2}\sup\operatorname{sech}^{2}\left[\frac{\sqrt{c}}{2}(\eta-c\tau^{\zeta})\right] \tag{16}$$

Its graph for $\zeta$ = ln2/ln3 is represented in fig. 2.
Secondly, we now consider the space-fractal Korteweg-de Vries equation:

$$\frac{\partial\Theta(\eta,\tau)}{\partial\tau}+6\Theta(\eta,\tau)\frac{\partial^{\zeta}\Theta(\eta,\tau)}{\partial\eta^{\zeta}}+\frac{\partial^{3\zeta}\Theta(\eta,\tau)}{\partial\eta^{3\zeta}}=0 \tag{17}$$

In view of the fractal complex transform:

$$\frac{\partial^{\zeta}\Theta[\mu(\eta),\tau]}{\partial\eta^{\zeta}}=\frac{\partial\Theta[\mu(\eta),\tau]}{\partial\mu}D^{(\zeta)}\eta^{\zeta} \tag{18}$$

$$\frac{\partial^{3\zeta}\Theta[\mu(\eta),\tau]}{\partial\eta^{3\zeta}}=\frac{\partial^{3}\Theta[\mu(\eta),\tau]}{\partial\mu^{3}}D^{(\zeta)}\eta^{\zeta} \tag{19}$$

where $\mu(\eta)=\eta^{\zeta}$, we obtain from eq. (17) the classical Korteweg-de Vries equation:

$$\frac{\partial\Theta(\eta,\mu)}{\partial\mu}+6\Theta(\eta,\mu)\frac{\partial\Theta(\eta,\mu)}{\partial\eta}+\frac{\partial^{3}\Theta(\eta,\mu)}{\partial\eta^{3}}=0 \tag{20}$$

The solution of eq. (20) can be written in the form [8]:

$$\Theta(\eta,\mu)=-\frac{c}{2}\operatorname{sech}^{2}\left[\frac{\sqrt{c}}{2}(\eta-c\mu)\right] \tag{21}$$

Thus, we have the non-differentiable solution of eq. (17), namely:

$$\Theta(\eta,\tau)=-\frac{c}{2}\sup\operatorname{sech}^{2}\left[\frac{\sqrt{c}}{2}(\eta^{\zeta}-c\tau)\right] \tag{22}$$

Its graph for $\zeta=\ln2/\ln3$ is illustrated in fig. 3.

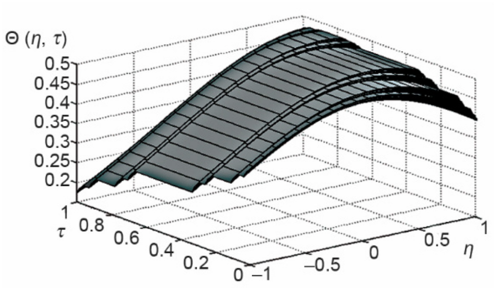


**Figure 2. The non-differentiable solution for $\zeta$ = ln2/ln3**

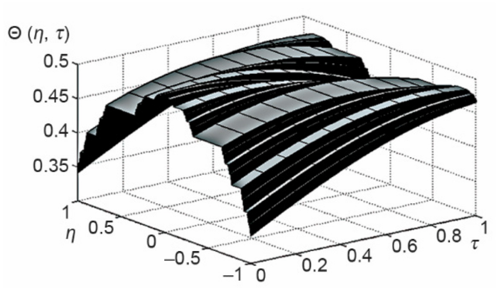


**Figure 3. The non-differentiable solution for $\zeta$ = ln2/ln3**

## Conclusion

In this work, we presented new applications of the fractal complex transform via a local fractional derivative to the time-fractal Korteweg-de Vries equation and space-fractal Korteweg-de Vries equation. The traveling wave solutions for the fractal Korteweg-de Vries equations are easily obtained with the help of the classical traveling wave solution of the Korteweg-de Vries equation.

## Nomenclature

$\eta$ – space co-ordinate, [m]
$\zeta$ – fractal dimensional order, [–]
$\tau$ – time, [s]

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