

A VARIATIONAL-PERTURBATION METHOD FOR SOLVING THE TIME-DEPENDENT SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

by

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In this paper, we combine the variational iteration method and perturbation theory to solve a time-dependent singularly perturbed reaction-diffusion problem. The problem is considered in the boundary layers and outer region. In the boundary layers, the problem is transformed by the variable substitution, and then the variational iteration method is employed to solve the transformed equation. In the outer region, we use the perturbation theory to obtain the approximation equation and the approximation solution. The final numerical experiments show that this method is accurate.

Key words: *reaction-diffusion problems, variational iteration method, perturbation theory, boundary layers*

Introduction

The reaction-diffusion equation is a kind of classical partial differential equation in fluid dynamics [1, 2]. The numerical methods [3-10] of singularly perturbed differential equations have received much attention. In this paper, we concern the singularly perturbed parabolic reaction-diffusion problem of the form:

$$\partial_t u(x,t) - \varepsilon \partial_x^2 u(x,t) + b(x)u(x,t) = f(x,t), \quad (x,t) \in (0,1) \times (0,T] \quad (1)$$

subject to the initial and boundary conditions:

$$u(x,0) = 0, \quad x \in [0,1] \quad (2)$$

$$u(0,t) = \phi(t), \quad u(1,t) = \psi(t), \quad t \in [0,T] \quad (3)$$

where ε is an arbitrary small parameter, such that $0 < \varepsilon \ll 1$, and $b(x) > 0$.

The classical numerical methods for solving singularly perturbed problems have a large amount of computation because they require extremely large number of mesh points to produce satisfactory numerical approximations. So the meshes are redesigned such that the standard finite difference methods [5] are still accurate. Meanwhile, researchers develop pa-

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parameter-robust numerical methods [6-9] whose accuracy can be guaranteed by the independent of the perturbation parameter for solving eq. (1). Sunil [10] gives an overlapping Schwarz domain decomposition method for solving singularly perturbed parabolic partial differential eq. (1). The numerical algorithm for eq. (1) is scant which inspires us to give a new algorithm for eq. (1).

To overcome the influence of the small parameter, the piecewise approximation solution is obtained in the boundary layers and outer region, respectively. In the boundary layers, the variable institution is introduced to expand the boundary layers, then the new equation with the variable institution is solved by the variational iteration method. In the outer region, the perturbation theory gives an asymptotic expansion solution for eq. (1), which is easy to be determined.

Variational-perturbation algorithm

Due to the perturbation parameter ε in eq. (1), we divide the interval $[0, 1]$ about x into three parts, namely, the boundary layers $[0, x_\varepsilon]$, $[1 - y_\varepsilon, 1]$, and the outer region $[x_\varepsilon, 1 - y_\varepsilon]$, where $0 < x_\varepsilon \ll 1$, $0 < y_\varepsilon \ll 1$ is the thickness of the boundary layer. Next, we will describe the variational-perturbation algorithm in the differential regions of x in detail.

In the boundary layers

When $(x, t) \in [0, x_\varepsilon] \times [0, T]$ or $(x, t) \in [1 - y_\varepsilon, 1] \times [0, T]$, eq. (1) is transformed by variable substitution, respectively.

(1) In $[0, x_\varepsilon] \times [0, T]$, putting $\bar{x} = x/\sqrt{\varepsilon}$, $U(\bar{x}, t) = u(\sqrt{\varepsilon}\bar{x}, t)$, then eq. (1) is converted into the form:

$$\partial_t U(\bar{x}, t) - \partial_{\bar{x}}^2 U(\bar{x}, t) + B_1(\bar{x})U(\bar{x}, t) = F_1(\bar{x}, t), \quad (\bar{x}, t) \in (0, \frac{x_\varepsilon}{\sqrt{\varepsilon}}) \times (0, T] \quad (5)$$

where $B_1(\bar{x}) = b(\sqrt{\varepsilon}\bar{x})$, $F_1(\bar{x}, t) = f(\sqrt{\varepsilon}\bar{x}, t)$, $U(\bar{x}, 0) = 0$.

(2) In $[1 - y_\varepsilon, 1] \times [0, T]$, putting $\bar{x} = (1 - x)/\sqrt{\varepsilon}$, $V(\bar{x}, t) = u(1 - \sqrt{\varepsilon}\bar{x}, t)$, then eq. (1) is transformed into the form:

$$\partial_t V(\bar{x}, t) - \partial_{\bar{x}}^2 V(\bar{x}, t) + B_2(\bar{x})V(\bar{x}, t) = F_2(\bar{x}, t), \quad (\bar{x}, t) \in (0, \frac{y_\varepsilon}{\sqrt{\varepsilon}}) \times (0, T] \quad (6)$$

where $B_2(\bar{x}) = b(1 - \sqrt{\varepsilon}\bar{x})$, $F_2(\bar{x}, t) = f(1 - \sqrt{\varepsilon}\bar{x}, t)$, $V(\bar{x}, 0) = 0$.

According to the variational iteration method [11-14], we can determine the Lagrange multiplier $\lambda = -1$, thus the iteration formulas of eqs. (5) and (6) are given by:

$$U_{n+1}(\bar{x}, t) = U_n(\bar{x}, t) - \int_0^t [\partial_s U_n(\bar{x}, s) - \partial_{\bar{x}}^2 U_n(\bar{x}, s) + B_1(\bar{x})U_n(\bar{x}, s) - F_1(\bar{x}, s)] ds \quad (7)$$

$$V_{n+1}(\bar{x}, t) = V_n(\bar{x}, t) - \int_0^t [\partial_s V_n(\bar{x}, s) - \partial_{\bar{x}}^2 V_n(\bar{x}, s) + B_2(\bar{x})V_n(\bar{x}, s) - F_2(\bar{x}, s)] ds \quad (8)$$

where $n = 0, 1, \dots$. To begin with $U_0(\bar{x}, t) = 0$, $V_0(\bar{x}, t) = 0$, one can obtain the approximation solution:

$$u^n(x,t) = \begin{cases} U_n\left(\frac{x}{\sqrt{\varepsilon}}, t\right), & (x,t) \in [0, x_\varepsilon] \times [0, T] \\ V_n\left(\frac{1-x}{\sqrt{\varepsilon}}, t\right), & (x,t) \in [1-y_\varepsilon, 1] \times [0, T] \end{cases}$$

In the outer region

When $(x,t) \in [x_\varepsilon, 1-y_\varepsilon] \times [0, T]$, using the perturbation technology [15], one has the asymptotic expansion of the solution in eq. (1):

$$u(x,t) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \dots \tag{9}$$

Taking eq. (9) into eq. (1), and comparing the coefficient of ε , we have:

$$\partial_t u_0(x,t) + b(x)\partial_x^2 u_0(x,t) = f(x,t), \quad (x,t) \in [x_\varepsilon, 1-y_\varepsilon] \times [0, T] \tag{10}$$

where $u_0(x,0) = 0$, then $u_0(x,t)$ is the approximation solution of eq. (1) in $[x_\varepsilon, 1-y_\varepsilon] \times [0, T]$. Equation (10) is a typical partial differential equation, so there are many numerical methods to obtain the approximation solution. Here we use command *NDSolve* in MATHEMATICA 8.0 to give numerical interpolation solution of eq. (10).

Numerical example

Considering the following time dependent singularly perturbed reaction-diffusion problem [10]:

$$\begin{cases} \partial_t u - \varepsilon \partial_x^2 u + [10 + 0.1 \sin(\pi x)]u = f & (x,t) \in (0,1) \times (0,1), \\ u(x,0) = 0, & x \in [0,1], \\ u(0,t) = \phi(t), \quad u(1,t) = \psi(t), & t \in [0,1], \end{cases}$$

where f, ϕ , and ψ are chosen so that the exact solution is:

$$u(x,t) = [1 - \exp(-t/100)](\exp(-x\sqrt{10/\varepsilon}) + \exp[-(1-x)\sqrt{10/\varepsilon}])$$

Here, we take $x_\varepsilon = y_\varepsilon = \sqrt{\varepsilon}$, $n = 3$. Table 1 displays the maximum errors for differential ε by the present method. We compare our numerical results with [10] in tab. 1, clearly, these results are well in accuracy.

Conclusion

In this paper, we give a new method to solving a singular perturbation partial differential equation by combing the variational iteration method and perturbation technology. For the perturbation parameter, we introduce the variable substitution to overcome the influence of small parameter for the problems. The algorithm is easy to implement, the final numerical results show that this algorithm is accurate.

Table 1. Comparison of maximum errors with [10]

$\varepsilon = 10^{-s}$	Our method	Ref. [10]
$s = 2$	1.55E-05	2.64E-05
$s = 3$	7.75E-06	2.65E-05
$s = 4$	6.16E-06	2.65E-05
$s = 5$	7.87E-06	2.65E-05
$s = 6$	6.58E-11	2.65E-05
$s = 7$	8.45E-06	2.65E-05
$s = 8$	8.48E-06	2.65E-05

Nomenclature

n - real number, [-]
 t - time, [s]
 x, \bar{x} - displacements, [m]
 x_ε - real number, [-]
 y_ε - real number, [-]

Greek symbols

ε - real number, [-]
 π - circumference ratio, [-]

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