LOCAL FRACTIONAL FUNCTIONAL DECOMPOSITION METHOD FOR SOLVING LOCAL FRACTIONAL POISSON EQUATION IN STEADY HEAT-CONDUCTION PROBLEM

by

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The steady heat-conduction problem via local fractional derivative is investigated in this paper. The analytical solution of the local fractional Poisson equation is obtained. The local fractional functional decomposition method is proposed to find the analytical solution of the partial differential equation in the steady heat--conduction problem.

Key words: *steady heat-conduction, Poisson equation, analytical solution, local fractional functional decomposition method, local fractional derivative*

Introduction

In this paper, we consider the local fractional Poisson equation in the steady heat-conduction problem [1]:

$$\frac{\partial^{2\beta}\Theta_\beta(x,y)}{\partial x^{2\beta}}+\frac{\partial^{2\beta}\Theta_\beta(x,y)}{\partial y^{2\beta}}=\sin_\beta(y^\beta) \tag{1}$$

subject to the initial-boundary value conditions:

$$\frac{\partial^{\beta}\Theta_\beta(0,y)}{\partial x^{\beta}}=\sin_\beta(y^\beta),\quad \Theta_\beta(0,y)=\sin_\beta(y^\beta),\quad \Theta_\beta(x,0)=\Theta_\beta(x,\pi)=0 \tag{2a,b,c}$$

where the local fractional derivative of $\Theta_\beta(x)$ at $x=x_0$ is given by [2-5]:

$$D_x^\beta\Theta_\beta(x_0)=\frac{\mathrm{d}^\beta}{\mathrm{d}x^\beta}\Theta_\beta(x)\Big|_{x=x_0}=\Theta_\beta^{(\beta)}(x)=\lim_{x\to x_0}\frac{\Delta^\beta[\Theta_\beta(x)-\Theta_\beta(x_0)]}{(x-x_0)^\beta} \tag{3}$$

with $\Delta^\beta[\Theta_\beta(x)-\Theta_\beta(x_0)]\cong\Gamma(1+\beta)\Delta[\Theta_\beta(x)-\Theta_\beta(x_0)]$.

Let $\Theta_\beta(x)$ be 2π-periodic. For $k\in Z$, the local fraction Fourier series of $\Theta_\beta(x)$ is given [6]:

$$\Theta_\beta(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\{a_n\cos_\beta[(kx)^\beta]+b_n\sin_\beta[(kx)^\beta]\} \tag{4}$$

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where the local fraction Fourier coefficients are:

$$a_n = \frac{2}{\pi^\beta}\int_0^\pi \Theta_\beta(x)\cos_\beta[(kx)^\beta](\mathrm{d}x)^\beta \tag{5a}$$

$$b_n = \frac{2}{\pi^\beta}\int_0^\pi \Theta_\beta(x)\sin_\beta[(kx)^\beta](\mathrm{d}x)^\beta \tag{5a}$$

with local fractional integral of $\phi_\beta(x)$ in the interval $[a,b]$, which is defined [4, 6]:

$${}_aI_b^{(\beta)}\,\phi_\beta(x) = \frac{1}{\Gamma(1+\beta)}\int_a^b \phi_\beta(t)(\mathrm{d}t)^\beta \tag{6}$$

The local fractional Laplace transform of $\phi(x)$ is given [6-8]:

$$\tilde{L}_\beta\{\phi(x)\} = \phi_s^{\tilde{L},\beta}(s) = \frac{1}{\Gamma(1+\beta)}\int_0^\infty E_\beta(-s^\beta x^\beta)\phi(x)(\mathrm{d}x)^\beta, \qquad 0<\beta\le 1 \tag{7a}$$

The inverse local fractional Laplace transform of $\phi(x)$ is given [6-8]:

$$\phi(x) = \tilde{L}_\beta^{-1}\{\phi_s^{L,\beta}(s)\} = \frac{1}{(2\pi)^\beta}\int_{\mu-i\infty}^{\mu+i\infty} E_\beta(s^\beta x^\beta)\phi_s^{\tilde{L},\beta}(s)(\mathrm{d}s)^\beta \tag{7b}$$

where $\phi(x)$ is local fractional continuous, $s^\beta = \mu^\beta + i^\beta\infty^\beta$, and $\mathrm{Re}(s)=\mu>0$.

The useful formula is listed [6]:

$$\tilde{L}_\beta\{\phi^{(2\beta)}(x)\} = s^{2\beta}\tilde{L}_\beta\{\phi(x)\} - s^\beta\phi(0) - \phi^{(\beta)}(0) \tag{7c}$$

The local fractional functional decomposition method was proposed in [9] and developed to handle the inhomogeneous wave equations [10]. In this paper, we use the local fractional functional decomposition method to solve the local fractional Poisson equation in the steady heat-conduction problem.

Solving local fractional Poisson equation in the steady heat-conduction problem

Following the local fractional functional decomposition method [9, 10], we consider the non-differentiable decomposition of the non-differentiable function systems $\{\sin_\beta[(ny)^\beta]\}_{n=0}^\infty$.

There are the functional coefficients of eqs. (1) and (2a-c), which are given:

$$\Theta_\beta(x,y) = \sum_{n=1}^\infty A_n(x)\sin_\beta[(ny)^\beta] \tag{8a}$$

$$\sin_\beta(y^\beta) = \sum_{n=1}^\infty \mathrm{B}_n\sin_\beta[(ny)^\beta] \tag{8b}$$

$$\sin_\beta(y^\beta) = \sum_{n=1}^\infty C_n\sin_\beta[(ny)^\beta] \tag{8c}$$

$$\sin_\beta(y^\beta)=\sum_{n=1}^{\infty}\mathrm{D}_n\sin_\beta[(ny)^\beta], \tag{8d}$$

where

$$A_n(x)=\frac{2}{\pi^\beta}\int_0^\pi \Theta_\beta(x,y)\sin_\beta[(ny)^\beta](\mathrm{d}y)^\beta \tag{9a}$$

$$B_n=\frac{2}{\pi^\beta}\int_0^\pi \sin_\beta(y^\beta)\sin_\beta[(ny)^\beta](\mathrm{d}y)^\beta \tag{9b}$$

$$C_n=\frac{2}{\pi^\beta}\int_0^\pi \sin_\beta(y^\beta)\sin_\beta[(ny)^\beta](\mathrm{d}y)^\beta \tag{9c}$$

$$D_n=\frac{2}{\pi^\beta}\int_0^\pi \sin_\beta(y^\beta)\sin_\beta[(ny)^\beta](\mathrm{d}y)^\beta\ . \tag{9d}$$

Thus, we have:

$$B_n=\begin{cases}0, \ \mathrm{n}\neq 1,\\ 1, \ \mathrm{n}{=}1,\end{cases}\quad C_n=\begin{cases}0, \ \mathrm{n}\neq 1,\\ 1, \ \mathrm{n}{=}1,\end{cases}\quad \mathrm{D}_n=\begin{cases}0, \ \mathrm{n}\neq 1,\\ 1, \ \mathrm{n}{=}1.\end{cases}$$

Submitting eqs. (8a-d) into eq. (1) gives:

$$\sum_{n=1}^{\infty}\frac{\partial^{2\beta}A_n(x)}{\partial x^{2\beta}}\sin_\beta[(ny)^\beta]+n^{2\beta}\sum_{n=1}^{\infty}A_n(x)\sin_\beta[(ny)^\beta]=\sum_{n=1}^{\infty}\mathrm{B}_n\sin_\beta[(ny)^\beta] \tag{10a}$$

which, for $n{=}1$, leads to:

$$\frac{\partial^{2\beta}A_1(x)}{\partial x^{2\beta}}+A_1(x)=1,\qquad \frac{\partial^{\beta}A_1(0)}{\partial x^{\beta}}=C_1=1,\qquad A_1(0)=D_1=1 \tag{10b,c,d}$$

Taking the local fractional Laplace transform gives:

$$s^{2\beta}A_1(s)-s^\beta-1+A_1(s)=1 \tag{11}$$

Therefore, we rewrite eq. (11):

$$A_1(s)=\frac{s^\beta+2}{s^{2\beta}+1} \tag{12}$$

Thus, taking inverse local fractional Laplace transform of eq. (12), we have:

$$A_1(x)=\cos_\beta(x^\beta)+2\sin_\beta(x^\beta) \tag{13}$$

Finally, we obtain the non-differentiable solution of eq. (1) given by:

$$\Theta_\beta(x,y) = [\cos_\beta(x^\beta) + 2\sin_\beta(x^\beta)]\sin_\beta(y^\beta) \tag{13}$$

Conclusion

In this work, we discussed the local fractional Poisson equation in the steady heat--conduction problem. The non-differentiable solution of the local fractional Poisson equation were obtained by using the local fractional functional decomposition method. The technology is very efficient to solve the partial differential equations in the steady heat-conduction problem.

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Nomenclature

x, y – space co-ordinates, [m]
β – fractal dimension, [–]
$\Theta\beta(x, y)$ – temperature, [K]

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