LOCAL FRACTIONAL VARIATIONAL ITERATION ALGORITHM III FOR THE DIFFUSION MODEL ASSOCIATED WITH NON-DIFFERENTIABLE HEAT TRANSFER

by

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This paper addresses a new application of the local fractional variational iteration algorithm III to solve the local fractional diffusion equation defined on Cantor sets associated with non-differentiable heat transfer.

Key words: *heat transfer, diffusion equation, analytical solution, variational iteration algorithm III, diffusion equation, local fractional calculus*

Introduction

Fractal heat-conduction problems were modelled by the local fractional differential equations [1-3]. The local fractional diffusion equation was used to describe the non-differentiable phenomena in fractal heat-conduction [4-10]. Many methods for diffusion defined on Cantor sets, such as the local fractional decomposition method [4, 5], functional method [6], variational iteration algorithm II [7], similarity method [8], differential transform method [9], homotopy perturbation method [10], and so on.

The theory of the local fractional variational iteration algorithms (namely, variational iteration algorithm I, variational iteration algorithm II, and variational iteration algorithm III) was used to solve the local fractional differential equations [2, 11-13]. The local fractional diffusion equation in the heat-conduction problem was written in the form [1, 12]:

$$\frac{\partial^{\xi}\psi_{\xi}(x,\tau)}{\partial\tau^{\xi}}=\kappa\frac{\partial^{2\xi}\psi_{\xi}(x,\tau)}{\partial x^{2\xi}} \tag{1}$$

where $\psi_{\xi}(x,\tau)$ is the temperature function with respect to the time τ and space x with the fractal dimension ξ $(0\leq\xi\leq1)$, κ – the fractal thermal diffusivity, and the local fractional partial derivative (LFPD) of $\mathfrak{A}(x)$ is defined [1]:

$$\frac{\partial^{\xi}\mathfrak{A}(x,\tau)}{\partial x^{\xi}}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\xi}[\mathfrak{A}(x,\tau)-\mathfrak{A}(x_0,\tau)]}{(x-x_0)^{\xi}} \tag{2}$$

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with

$$\Delta^{\xi}[\mathfrak{A}(x,\tau)-\mathfrak{A}(x_0,\tau)]\cong\Gamma(1+\xi)[\mathrm{H}(x,\tau)-\mathrm{H}(x_0,\tau)] \tag{3}$$

In this paper, our aim is to solve the local fractional diffusion equation in the heat--conduction problem by using the local fractional variational iteration algorithm III.

The local fractional variational iteration algorithm III

In order to present the technology, we rewrite eq. (1) in the form:

$$L_{\xi}\psi_{\xi}(x,\tau)=\kappa\frac{\partial^{2\xi}\psi_{\xi}(x,\tau)}{\partial x^{2\xi}} \tag{4}$$

According to the theory of the local fractional variational iteration algorithm III, we write the correction functional given by:

$$\psi_{\xi,n+2}(x,\tau)=\psi_{\xi,n+1}(x,\tau)+{}_0I_{\tau}^{(\zeta)}\left\{\lambda\kappa\frac{\partial^{2\xi}}{\partial x^{2\xi}}[\widetilde{\psi}_{\xi,n+1}(x,\tau)-\widetilde{\psi}_{\xi,n}(x,\tau)]\right\} \tag{5}$$

where ${}_0I_{\tau}^{(\zeta)}$ is the local fractional integral operator [1-2,12] and $\widetilde{\psi}_{\xi,n}$ is considered as a restricted local fractional variation, *i. e.* $\delta^{\xi}\widetilde{\psi}_{\xi,n}=0$.

The local fractional variational iteration algorithm III is structured in the form:

$$\psi_{\xi,n+2}(x,\tau)=\psi_{\xi,n+1}(x,\tau)+{}_0I_{\tau}^{(\zeta)}\left\{\lambda\kappa\frac{\partial^{2\xi}}{\partial x^{2\xi}}[\psi_{\xi,n+1}(x,\tau)-\psi_{\xi,n}(x,\tau)]\right\} \tag{6}$$

where λ is the identified fractal Lagrange multiplier.

According to variational iteration algorithm I [12], we have:

$$\lambda=-1 \tag{7}$$

such that eq. (7) can be rewrite in the form:

$$\psi_{\xi,n+2}(x,\tau)=\psi_{\xi,n+1}(x,\tau)+{}_0I_{\tau}^{(\zeta)}\left\{\kappa\frac{\partial^{2\xi}}{\partial x^{2\xi}}[\widetilde{\psi}_{\xi,n+1}(x,\tau)-\widetilde{\psi}_{\xi,n}(x,\tau)]\right\} \tag{8}$$

Solving the local fractional diffusion equation in the heat-conduction problem

We consider the initial condition of eq. (1), which is given by:

$$\psi_{\xi}(x,0)=E_{\xi}(x^{\xi}) \tag{9}$$

Following eq. (8), we structure the following iterative formula:

$$\begin{cases} \psi_{\xi,n+2}(x,\tau) = \psi_{\xi,n+1}(x,\tau) + {}_0I_\tau^{(\zeta)} \left\{ \kappa \dfrac{\partial^{2\xi}}{\partial x^{2\xi}} (\tilde{\psi}_{\xi,n+1}[x,\tau) - \tilde{\psi}_{\xi,n}(x,\tau)] \right\} \\ \psi_{\xi,1}(x,\tau) = E_\xi(x^\xi)\left[1 + \dfrac{\kappa\tau^\xi}{\Gamma(1+\xi)}\right] \\ \psi_{\xi,0}(x,\tau) = E_\xi(x^\xi) \end{cases} \tag{10}$$

In view of eq. (10), the approximate solutions are:

$$\begin{aligned} \psi_{\xi,2}(x,\tau) &= E_\xi(x^\xi)\left[1 + \frac{\kappa\tau^\xi}{\Gamma(1+\xi)}\right] + {}_0I_\tau^{(\zeta)} \left\{ \kappa \frac{\partial^{2\xi}}{\partial x^{2\xi}} \left[E_\xi(x^\xi) \frac{\kappa\tau^\xi}{\Gamma(1+\xi)} \right] \right\} \\ &= E_\xi(x^\xi) \sum_{i=0}^{2} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} \end{aligned} \tag{11}$$

$$\begin{aligned} \psi_{\xi,3}(x,\tau) &= E_\xi(x^\xi) \sum_{i=0}^{2} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} + {}_0I_\tau^{(\zeta)} \left\{ \kappa \frac{\partial^{2\xi}}{\partial x^{2\xi}} \left[E_\xi(x^\xi) \frac{\kappa^2\tau^{2\xi}}{\Gamma(1+2\xi)} \right] \right\} \\ &= E_\xi(x^\xi) \sum_{i=0}^{3} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} \end{aligned} \tag{12}$$

$$\begin{aligned} \psi_{\xi,4}(x,\tau) &= E_\xi(x^\xi) \sum_{i=0}^{3} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} + {}_0I_\tau^{(\zeta)} \left\{ \kappa \frac{\partial^{2\xi}}{\partial x^{2\xi}} \left[E_\xi(x^\xi) \frac{\kappa^3\tau^{3\xi}}{\Gamma(1+3\xi)} \right] \right\} \\ &= E_\xi(x^\xi) \sum_{i=0}^{4} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} \end{aligned} \tag{13}$$

and so on.

Finally, we obtain:

$$\psi_\xi(x,\tau) = \lim_{n\to\infty} \psi_{\xi,n}(x,\tau) = E_\xi(x^\xi) \sum_{i=0}^{\infty} \frac{\kappa^j \tau^{i\xi}}{\Gamma(1+i\xi)} = E_\xi(x^\xi) E_\xi(\kappa\tau^\xi) \tag{14}$$

Conclusion

We investigated the local fractional diffusion equation in the heat-conduction problem by using the local fractional variational iteration algorithm III. We structured the iterative formulas via local fractional calculus. We efficiently used the local fractional variational iteration algorithm to find the non-differentiable solution for the fractal heat-conduction problem.

Nomenclature

x – space co-ordinate, [m]
${}_0I_\tau^{(\zeta)}$ – local fractional integral operator, [–]
$\psi_\xi(x, \tau)$ – temperature field, [K]
$\partial^\xi/\partial x^\xi$ – LFPD of ϑ, [–]
ξ – fractal dimensional order, [–]

References

[1] ***, *Fractional Dynamics* (Eds. C. Cattani, H. M. Srivastava, X.-J. Yang), De Gruyter Open, Berlin, 2015, ISBN 978-3-11-029316-6
[2] Zhang, Y., *et al.*, Local Fractional Variational Iteration Algorithm II for Non-Homogeneous Model Associated with the Non-Differentiable Heat Flow, *Advances in Mechanical Engineering*, *7* (2015), 10, pp. 1-7
[3] Zhang, Y., *et al.*, Local Fractional Homotopy Perturbation Method for Solving Non-Homogeneous Heat Conduction Equations in Fractal Domains, *Entropy*, *17* (2015), 10, pp. 6753-6764
[4] Yang, X. J., *et al.*, Approximate Solutions for Diffusion Equations on Cantor Space-Time, *Proceedings,* Romanian Academy, Series A, 201, Vol. 14, No. 2, pp. 127-133
[5] Yang, X. J., *et al.*, Fractal Boundary Value Problems for Integral and Differential Equations with Local Fractional Operators, *Thermal Science*, *19* (2015), 3, pp. 959-966
[6] Cao, Y., *et al.*, Local Fractional Functional Method for Solving Diffusion Equations on Cantor Sets, *Abstract Applied Analysis*, *2014* (2014), ID 803696
[7] Yang, Z., *et al.*, A New Iteration Algorithm for Solving the Diffusion Problem in Non-Differentiable Heat Transfer, *Thermal Science*, *19* (2015), Suppl. 1, pp. S105-S108
[8] Yang, X. J., *et al.*, Local Fractional Similarity Solution for the Diffusion Equation Defined on Cantor Sets, *Applied Mathematical Letters*, *47* (2015), Mar., pp. 54-60
[9] Yang, X. J., *et al.*, A New Numerical Technique for Solving the Local Fractional Diffusion Equation: Two-Dimensional Extended Differential Transform Approach, *Applied Mathematics and Computation*, *274* (2016), 1, pp. 143-151
[10] Yang, X.-J., *et al.*, Local Fractional Homotopy Perturbation Method for Solving Fractal Partial Differential Equations Arising in Mathematical Physics, *Romanian Reports in Physics*, *67* (2015), 3, pp. 752-761
[11] Yang, X. J., Local Fractional Kernel Transform in Fractal Space and Its Applications, *Advances in Computational Mathematics and its Applications*, *1* (2012), 2, pp. 86-93
[12] Yang, X. J., Baleanu, D., Fractal Heat Conduction Problem Solved by Local Fractional Variation Iteration Method, *Thermal Science*, *17* (2013), 2, pp. 625-628
[13] Liu, H. Y., *et al.*, Fractional Calculus for Nanoscale Flow and Heat Transfer, *International Journal of Numerical Methods for Heat & Fluid Flow*, *24* (2014), 6, pp. 1227-1250

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