A NEW COMPUTATIONAL METHOD FOR FRACTAL HEAT-DIFFUSION VIA LOCAL FRACTIONAL DERIVATIVE

by

Geng-Yuan LIU [a,b]

[a] State Key Joint Laboratory of Environment Simulation and Pollution Control, School of Environment, Beijing Normal University, Beijing, China
[b] Beijing Engineering Research Center for Watershed Environmental Restoration and Integrated Ecological Regulation, Beijing, China

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The fractal heat-conduction problem via local fractional derivative is investigated in this paper. The solution of the fractal heat-diffusion equation is obtained. The characteristic equation method is proposed to find the analytical solution of the partial differential equation in fractal heat-conduction problem.

Key words: *heat-diffusion equation, characteristic equation method, analytical solution, local fractional derivative*

Introduction

Fractal heat-transfer problems were described by the kernel functions of differentiable and non-differentiable types via the fractional calculus and local fractional calculus. For example, the fractional heat-transfer [1, 2] and heat-conduction equations [3-5] via fractional derivative were presented. With the help of the implicit difference method [5], heat integral-balance method [6], finite-difference method [7], the differentiable solutions for fractional heat diffusion equations were reported. The heat-diffusion [8-14] and heat-transfer [15] equations in fractal media via local fractional derivative were presented. Several methods for finding the non-differentiable solutions for fractal heat-conduction problems, *e. g.* the local fractional variation iteration algorithm I [16] and algorithm II [17], local fractional Fourier series [18], Adomian decomposition [19-21], Laplace decomposition [22], homotopy perturbation [23], similarity variable [24], and characteristic equation [25] methods, were developed. The aim of the paper is to propose the characteristic equation method (CEM) [25] to solve the fractal heat-diffusion equation via local fractional derivative.

Mathematical tools

In this section, we present the concept of the local fractional derivative and the local fractional derivatives of the non-differentiable functions.

Definition 1. The local fractional derivative of the function $\Theta(\vartheta)$ of order $\theta\,(0<\theta<1)$ at $\vartheta=\vartheta_0$ is defined by [10-18]:

$$D_{\vartheta}^{(\theta)}\Theta(\vartheta_0)=\frac{\mathrm{d}^{\theta}\Theta(\vartheta)}{\mathrm{d}\vartheta^{\theta}}\Big|_{\vartheta=\vartheta_0}=\lim_{\vartheta\to\vartheta_0}\frac{\Delta^{\theta}[\Theta(\vartheta)-\Theta(\vartheta_0)]}{(\vartheta-\vartheta_0)^{\theta}} \tag{1}$$

* Author's e-mail: dliugengyuan@163.com

Table 1. The prosperities of the local fractional derivatives of the special functions

Special functions	Local fractional derivatives
$\dfrac{\vartheta^{k\theta}}{\Gamma(1+k\theta)}\ (k \in \mathbb{N})$	$\dfrac{\vartheta^{(k-1)\theta}}{\Gamma[1+(k-1)\theta]}$
$E_\theta(k\vartheta^\theta)(k \in \mathbb{R})$	$kE_\theta(k\vartheta^\theta)$

where $\Delta^\theta[\Theta(\vartheta)-\Theta(\vartheta_0)] \cong \Gamma(1+\theta)\Delta[\Theta(\vartheta)-\Theta(\vartheta_0)]$.

The prosperities of the local fractional derivatives are listed in tab. 1.

Definition 2. The local fractional partial derivative of the function $\Theta(\vartheta,\Lambda)$ of order $\theta\,(0<\theta<1)$ at $(\vartheta,\Lambda)=(\vartheta_0,\Lambda_0)$ is defined as [10, 12]:

$$\frac{\partial^\theta \Theta(\vartheta,\Lambda_0)}{\partial \vartheta^\theta}\bigg|_{(\vartheta_0,\Lambda_0)} = \lim_{\vartheta\to\vartheta_0} \frac{\Delta^\theta[\Theta(\vartheta,\Lambda_0)-\Theta(\vartheta_0,\Lambda_0)]}{(\vartheta-\vartheta_0)^\theta} \tag{2}$$

where $\Delta^\theta[\Theta(\vartheta,\Lambda_0)-\Theta(\vartheta_0,\Lambda_0)] \cong \Gamma(1+\theta)\Delta[\Theta(\vartheta,\Lambda_0)-\Theta(\vartheta_0,\Lambda_0)]$.

The local fractional partial derivative of the function $\Theta(\vartheta,\Lambda)$ of order $2\theta\,(0<\theta<1)$ is expressed by:

$$\frac{\partial^\theta}{\partial \vartheta^\theta}\left[\frac{\partial^\theta \Theta(\vartheta,\Lambda)}{\partial \vartheta^\theta}\right] = \frac{\partial^{2\theta}\Theta(\vartheta,\Lambda)}{\partial \vartheta^{2\theta}} \tag{3}$$

Solving the fractal heat-diffusion equation

The fractal heat-diffusion equation [10, 12, 14] via local fractional derivative is written in the form:

$$\frac{\partial^\theta \Theta(\vartheta,\tau)}{\partial \tau^\theta} - \kappa \frac{\partial^{2\theta}\Theta(\vartheta,\tau)}{\partial \vartheta^{2\theta}} = 0 \tag{4}$$

where κ is a heat-diffusive coefficient.

By assuming the non-differentiable solution of eq. (4):

$$\Theta(\vartheta,\tau) = E_\theta(\rho\tau^\theta)E_\theta(\sigma\vartheta^\theta) \tag{5}$$

the characteristic value takes the form:

$$\rho - \kappa\sigma^2 = 0 \tag{6}$$

With the help of eqs. (5) and (6), the solution of non-differentiable type via Mittag--Leffler function defined on Cantor set is written:

$$\Theta(\vartheta,\tau) = \nu E_\theta(\kappa\sigma^2\tau^\theta)E_\theta(\sigma\vartheta^\theta) \tag{7}$$

where ν is a constant, and the corresponding graph is represented in the fig. 1.

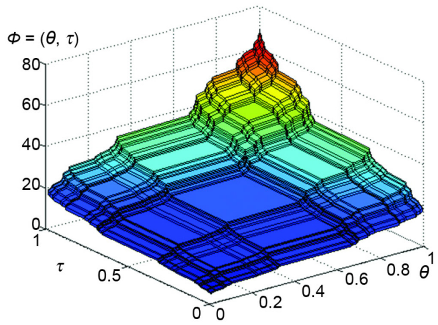

Figure 1. The solution of non-differentiable type of eq. (4) when $\nu = 1$, $\kappa = 1$, and $\sigma = 1$ *(for color image see journal web-site)*

Conclusions

In this work, the fractal heat-diffusion equation via local fractional derivative has been investigated. The CEM was employed to find the non-differentiable solutions of the fractal heat-transfer problems. The procedures of the CEM are listed:

- set the special solution of the local fractional partial differential equation,
- find out the characteristic value, and
- get the general solution of the partial differential equation.

The fractal heat-diffusion equation in the fractal dimension $\theta = \ln 2/\ln 3$ was discussed. The obtained result was proposed to illustrate the non-differentiable behaviour of the fractal heat-conduction.

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Nomenclature

ϑ – space co-ordinate, [m]
θ – fractal order, [–]
$\Theta(\vartheta, \tau)$ – concentration, [–]
τ – time, [s]

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